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Cooperative Control for Teams on Communication Graphs

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Cooperative Control for Teams on Communication Networks

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Examples from nature



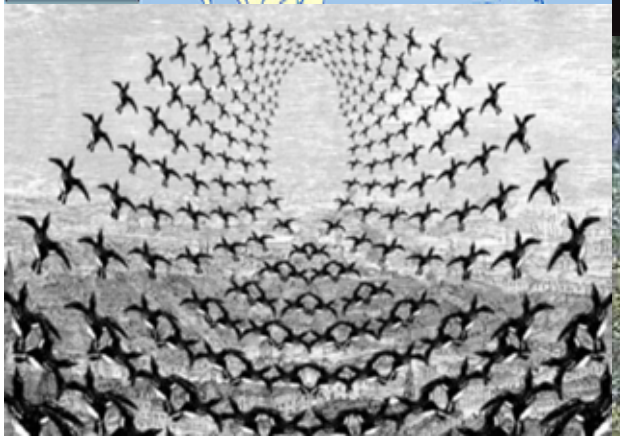




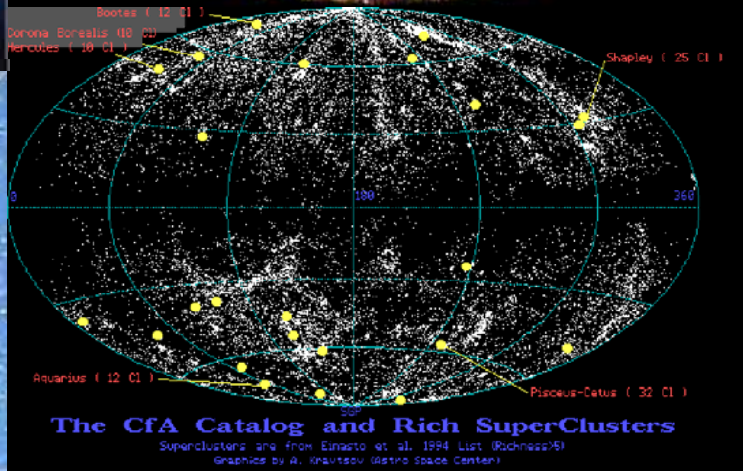




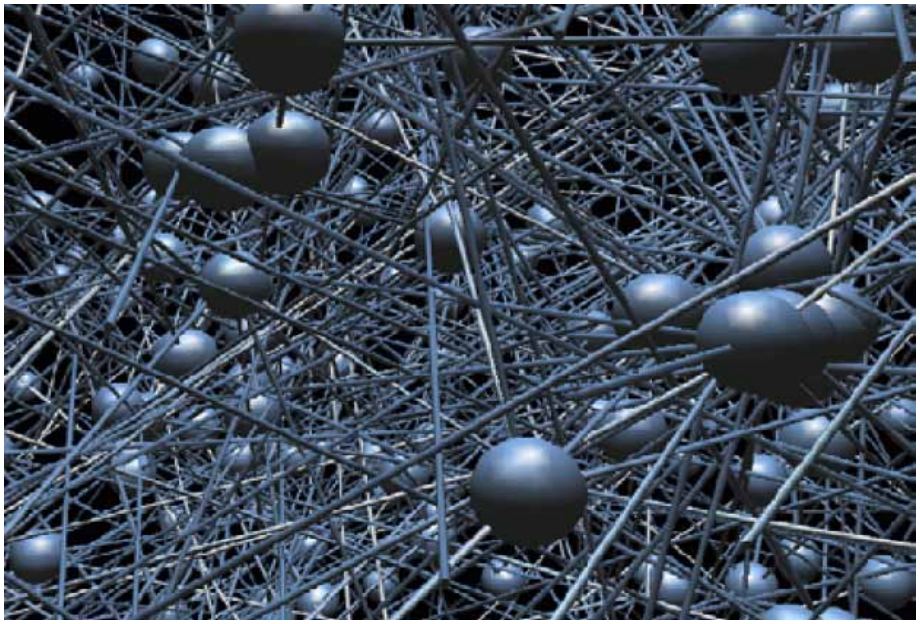


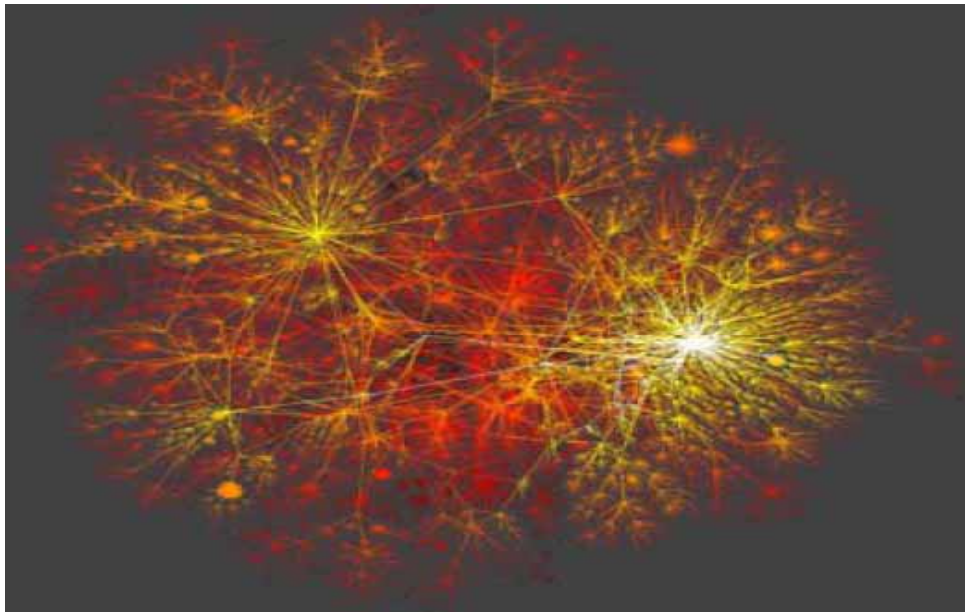


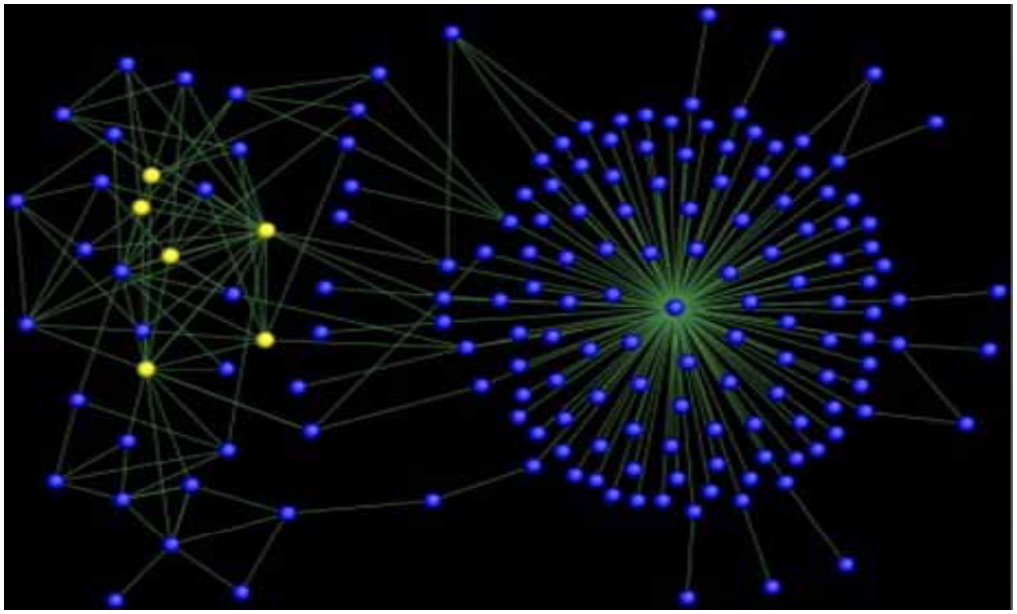




Natural and biological structures

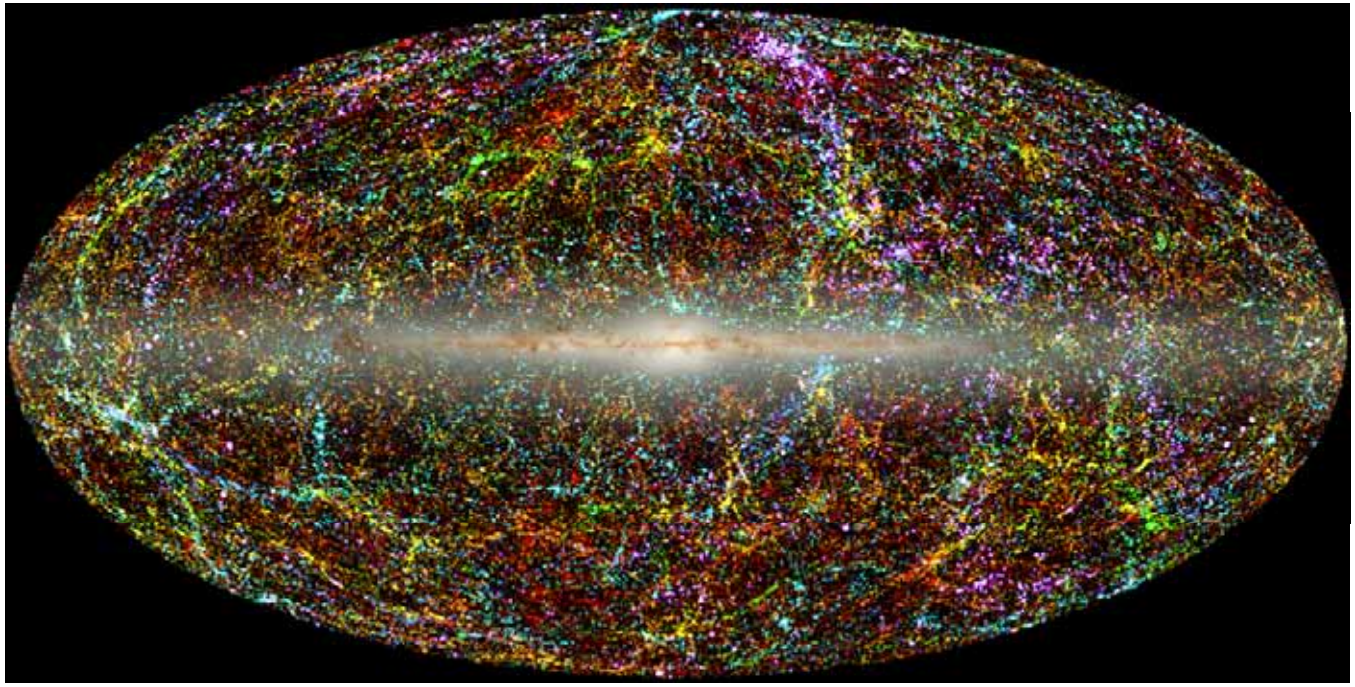




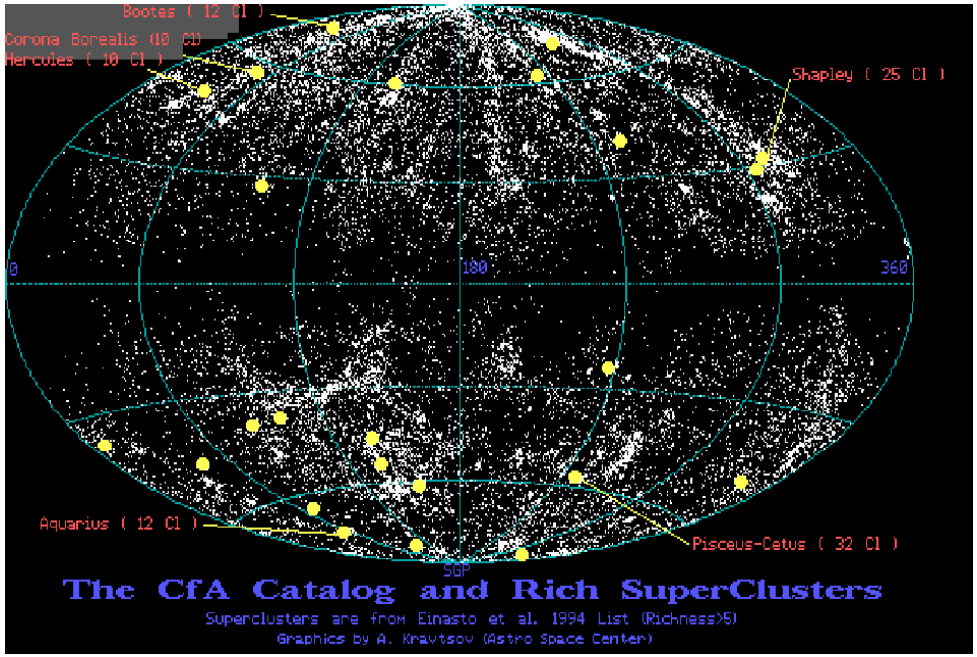


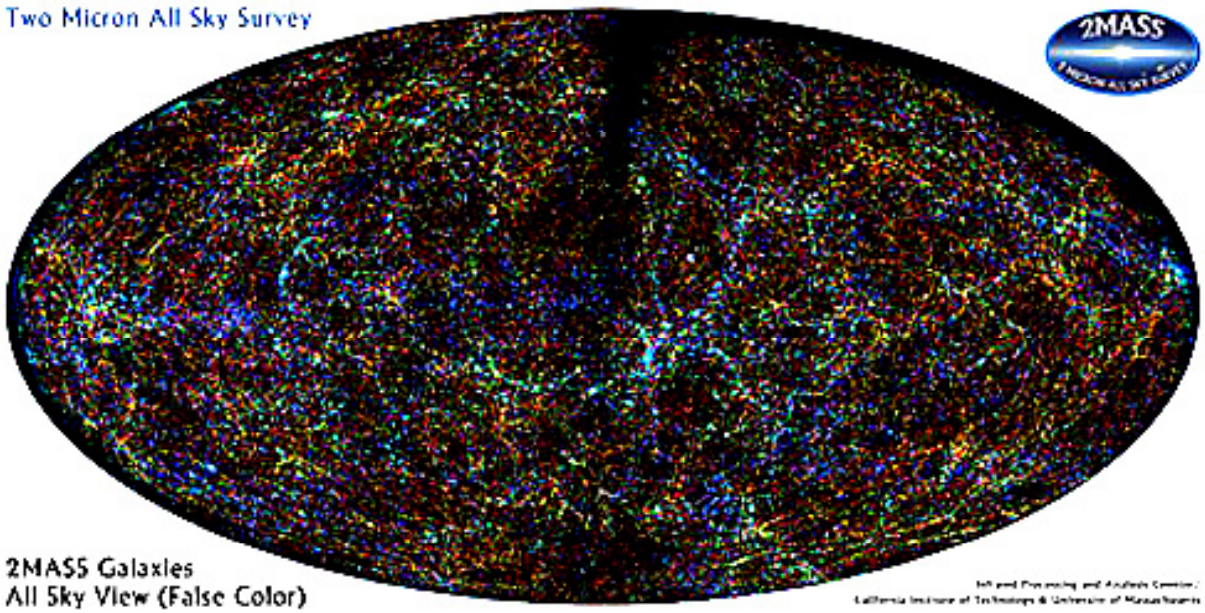
Many of the beautiful pictures are from a lecture by **Ron Chen**, City U. Hong Kong Pinning Control of Graphs

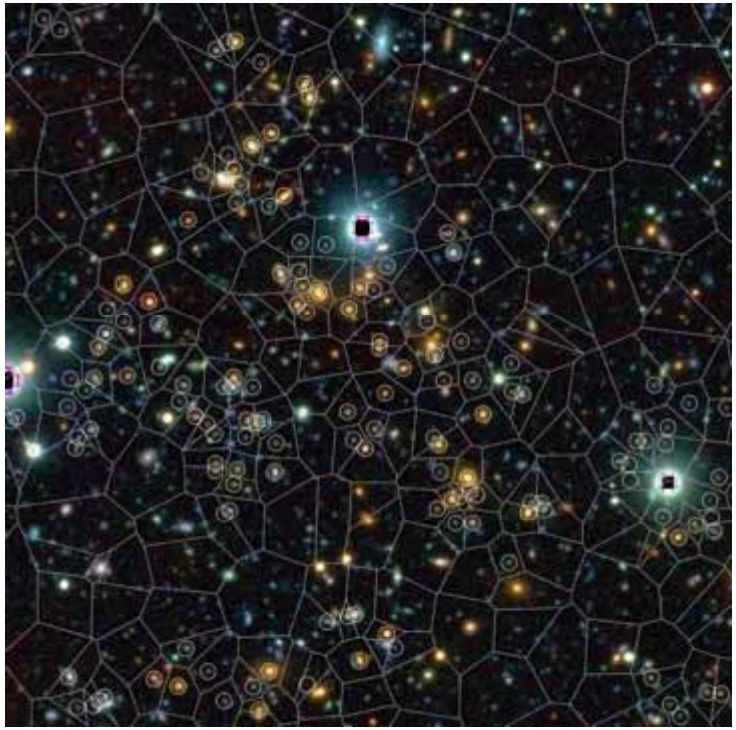
Distribution of galaxies in the universe

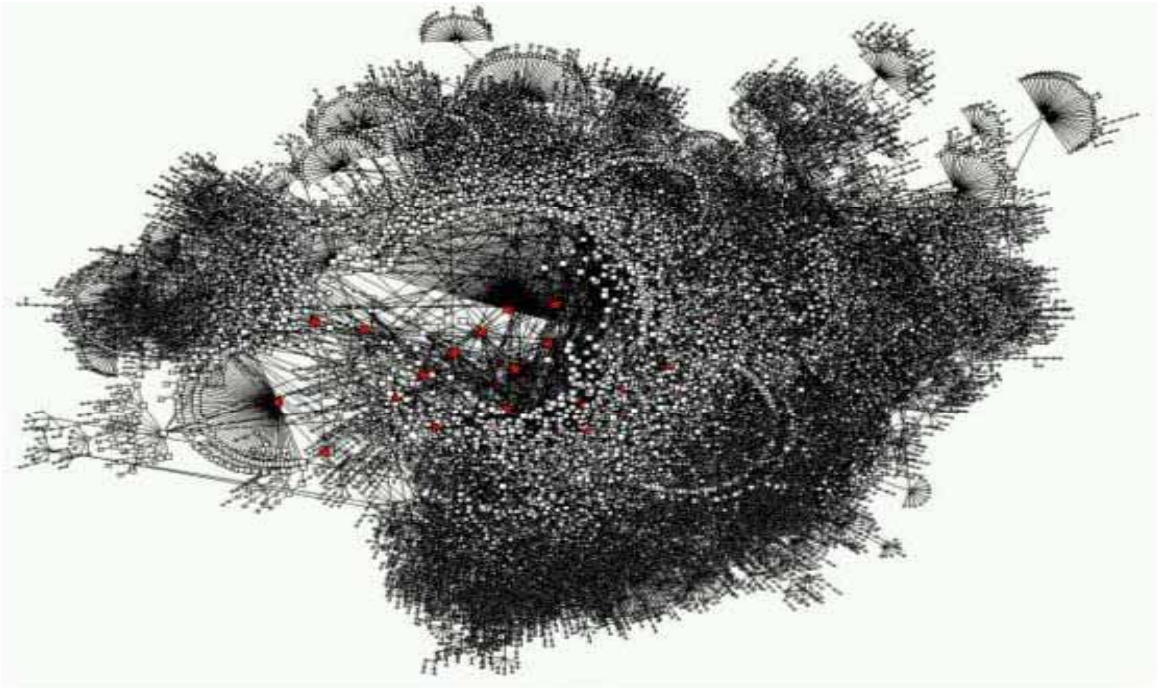







Two Micron All Sky Survey
2MASS
2MASS Galaxies
All Sky View (False Color)





The internet

(a)



(b)



(c)



(d)

ecosystem

Professional
Collaboration network

Barcelona rail network

J.J. Finnigan, Complex science for a complex world

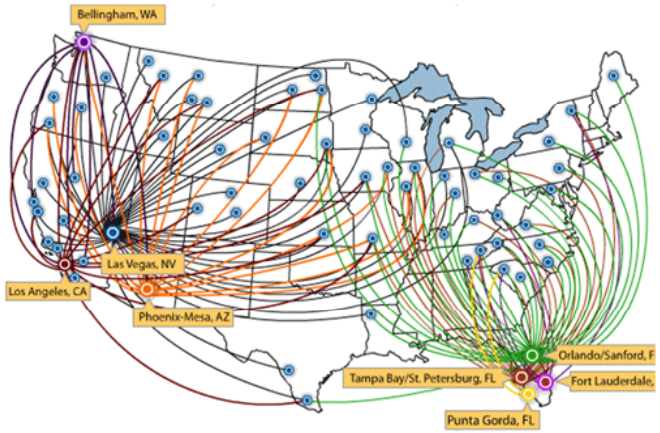
Airline Route Systems







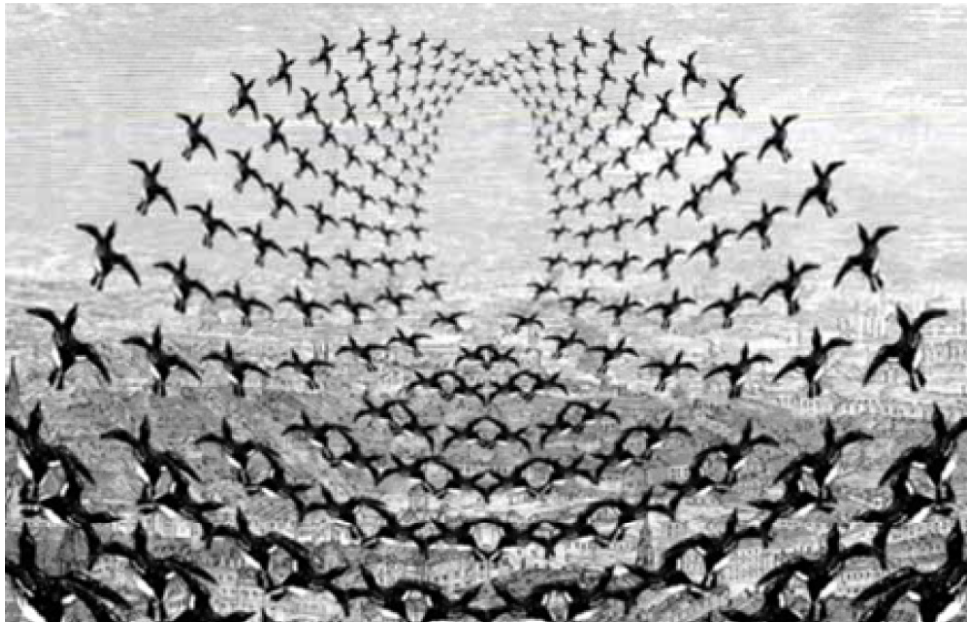



Bellingham, WA
Las Vegas, NV
Los Angeles, CA
Phoenix-Mesa, AZ
Orlando/Sanford, F
Tampa Bay/St. Petersburg, FL
Fort Lauderdale,
Punta Gorda, FL

Motions of biological groups



Fish school



Birds flock



Locusts swarm



Fireflies synchronize

Herd and Panic Behavior During Emergency Building Egress

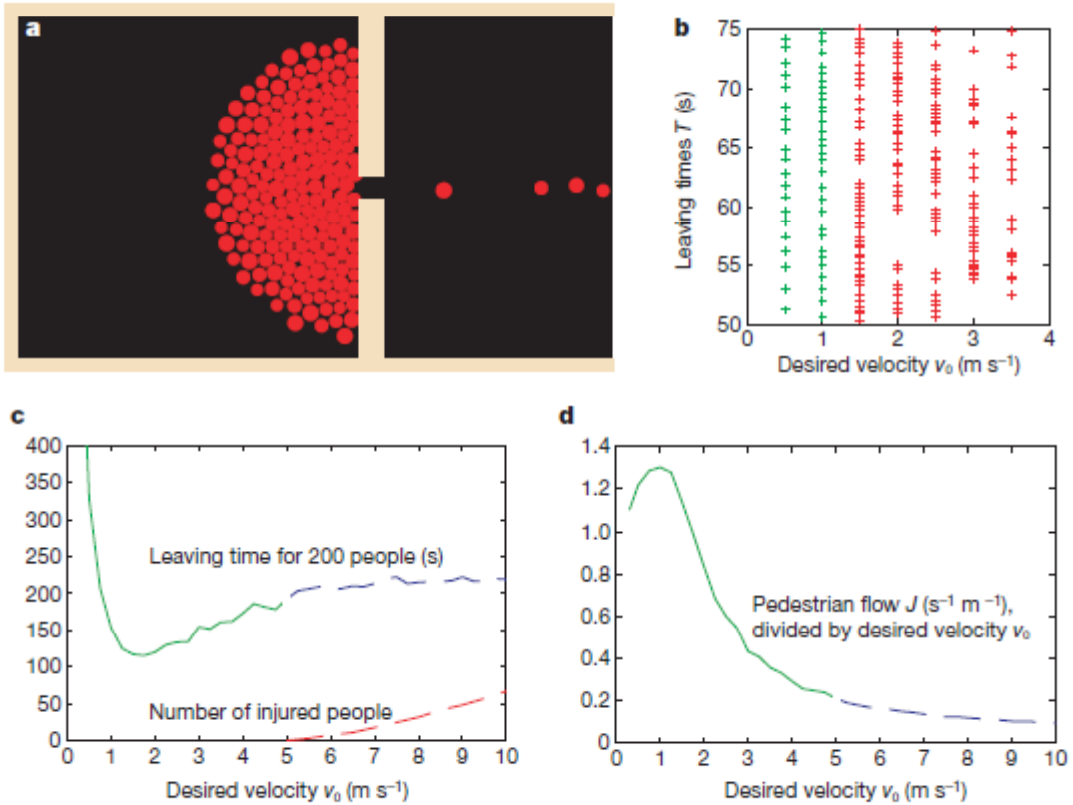


Figure 1 Simulation of pedestrians moving with identical desired velocity $v_i^0 = v_0$ towards the 1-m-wide exit of a room of size 15 m × 15 m. **a**, Snapshot of the simulation. Dynamic simulations are available at http://angel.elte.hu/~panic/. **b**, Leaving times of pedestrians for various desired velocities v_0. Irregular outflow due to clogging is observed for high desired velocities ($v_0 \geq 1.5\,\mathrm{m\,s^{-1}}$, red plus signs). **c**, Under conditions of normal walking, the time for 200 pedestrians to leave the room decreases with growing v_0. Desired velocities higher than $1.5\,\mathrm{m\,s^{-1}}$ reduce the efficiency of leaving, which becomes particularly clear when the outflow J is divided by the desired velocity (**d**). This is due to pushing, which causes additional friction effects. Moreover, above a desired velocity of about $v_0 = 5\,\mathrm{m\,s^{-1}}$ (corresponding to dashed lines in **c** and **d**) people are injured and become non-moving obstacles for others, if the sum of the magnitudes of the radial forces acting on them divided by their circumference exceeds a pressure of $1{,}600\,\mathrm{N\,m^{-1}}$ (ref. 5). Owing to the above 'faster-is-slower effect', panics can be triggered by pedestrian counterflows[2], which cause delays to the crowd intending to leave. This makes the stopped pedestrians impatient and pushy which may be described by increasing the desired velocity according to $v_i^0(t) = [1 - p_i(t)]v_i^0(0) + p_i(t)v_i^{\max}$, where $v_i^0(0)$ is the initial, and $v_i^{\max}$ the maximum desired velocity. The time-dependent parameter $p_i(t) = 1 - \bar{v}_i(t)/v_i^0$, where $\bar{v}_i(t)$ denotes the average speed in the desired direction of motion, is a measure of impatience. Altogether, long waiting times increase the desired velocity, which can produce inefficient outflow. This further increases the waiting times, and so on, so that this tragic feedback can eventually trigger panics. It is therefore imperative to have sufficiently wide exits and to prevent counterflows when big crowds want to leave.

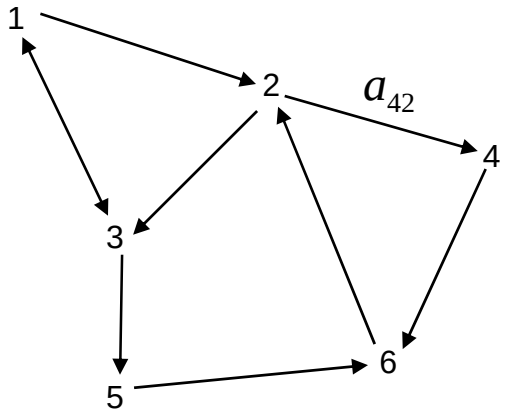
Helbring, Farkas, Vicsek, Nature 2000



Communication Graph

(V,E)

N nodes



Adjacency matrix

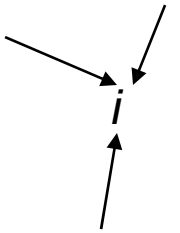
$$A = [a_{ij}]$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

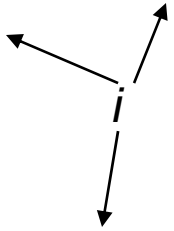
$$a_{ij} > 0 \; if \; (v_j, v_i) \in E$$

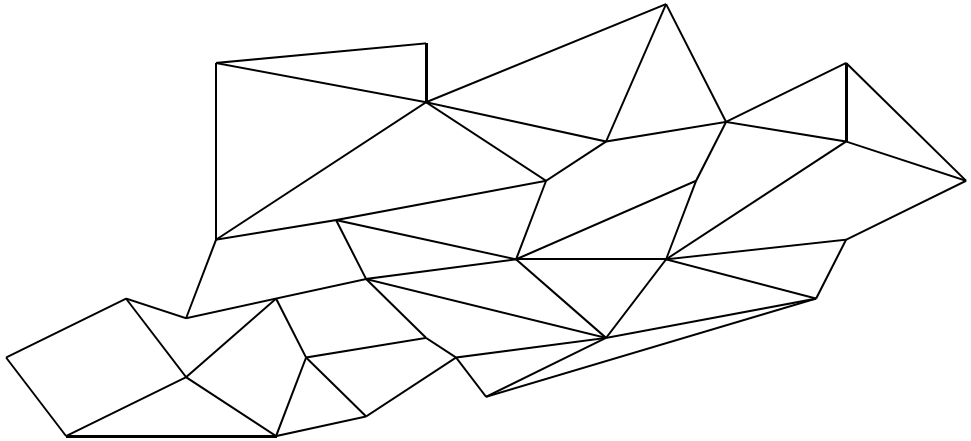
$$if \;\; j \in N_i$$

$d_i = \sum_{j=1}^{N} a_{ij}$ Row sum= in-degree N_i In-neighbors of node i



$d_i^o = \sum_{j=1}^{N} a_{ji}$ Col sum= out-degree N_o Out-neighbors of node i



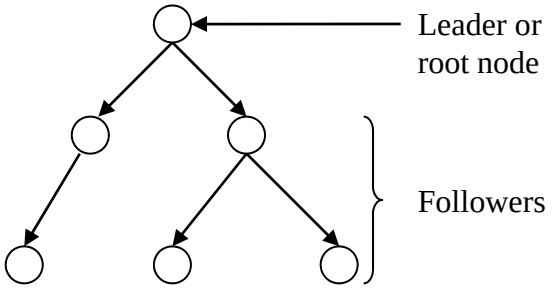


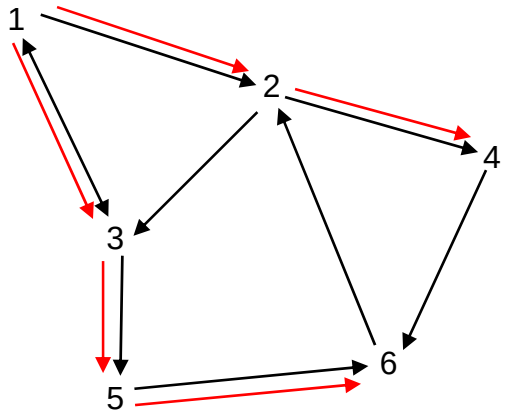
Strongly connected if for all nodes *i* and *j* there is a path from *i* to *j*.

Diameter= length of longest path between two nodes

Volume = sum of in-degrees $Vol = \sum_{i=1}^{N} d_i$

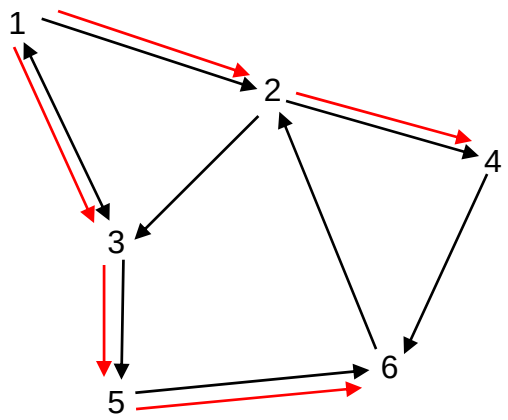
Tree- every node has in-degree=1





Spanning tree

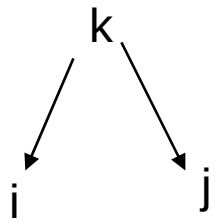
Spanning node



Exists a spanning tree iff there is a node having a path to all other nodes

Graph strongly connected implies exists a spanning tree

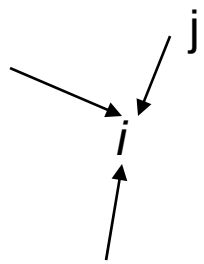
Quasi-strongly connected if for all nodes *i* and *j*
there exists a node *k* with a path to *i* and a path to *j*



Exists a spanning tree iff quasi-strongly connected

Dynamic Graph

Each node has an associated state $\dot{x}_i = u_i$



Standard local voting protocol $u_i = \sum_{j \in N_i} a_{ij}(x_j - x_i)$

$$u_i = -x_i \sum_{j \in N_i} a_{ij} + \sum_{j \in N_i} a_{ij} x_j = -d_i x_i + \begin{bmatrix} a_{i1} & \cdots & a_{iN} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$$

x

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} \qquad D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_N \end{bmatrix}$$

$u = -Dx + Ax = -(D - A)x = -Lx$ L=D-A = graph Laplacian matrix

$\dot{x} = -Lx$ Closed-loop dynamics

If x is an n-vector then $\dot{x} = -(L \otimes I_n)x$

Closed-loop system with local voting protocol

$$\dot{x} = -Lx$$

Modal decomposition $x(t) = e^{-Lt}x(0) = \sum_{j=1}^{N} w_i^T e^{-\lambda_i t} v_i x(0) = \sum_{j=1}^{N} \left(w_i^T x(0) \right) e^{-\lambda_i t} v_i$

Closed-loop dynamics depends on eigenstructure of L

v_i = right e-vectors
w_i = left e-vectors

$$L = D - A = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_N \end{bmatrix} - \begin{bmatrix} 0 & a_{12} & \cdots & a_{1N} \\ a_{21} & 0 & & \vdots \\ \vdots & & \ddots & \\ a_{N1} & \cdots & & 0 \end{bmatrix} \quad , \quad d_i = \sum_{j=1}^{N} a_{ij} \text{ = row sum}$$

L has row sum of 0

$$0 = L \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = L\underline{1} \qquad \text{or} \qquad 0 = -L\underline{1} = (\lambda_1 I - L) v_1$$

so $\lambda_1 = 0$ and $v_1 = \underline{1}$ is the right e-vector

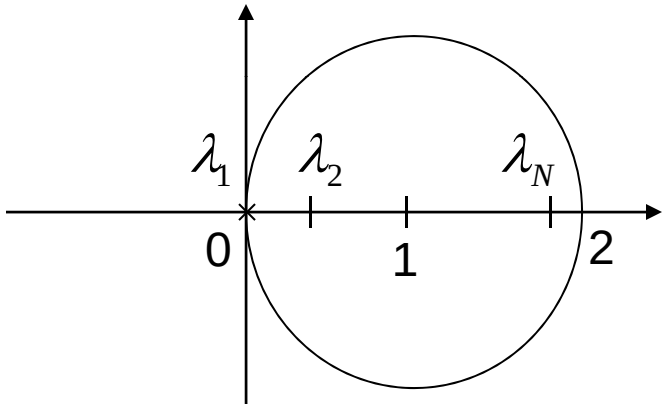
Thm. Gerschgorin Circle Criterion. All eigenvalues of matrix $E = [e_{ij}] \in R^{n \times n}$

are located within the following union of *n* discs

$$\bigcup_{i=1}^{n} \left\{ z \in C : |z - e_{ii}| \le \sum_{j \ne i} |e_{ij}| \right\}$$

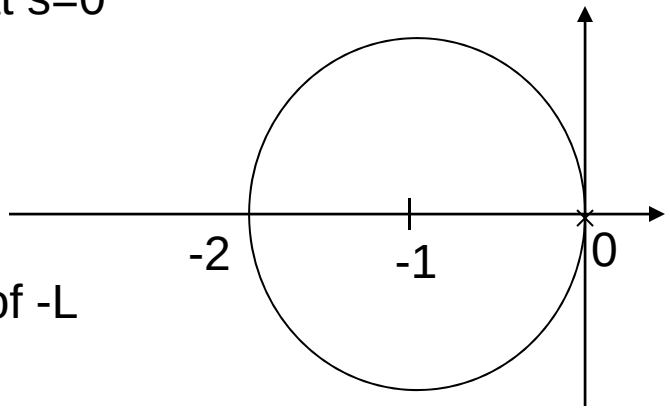
Normalized Laplacian $D^{-1}L = D^{-1}(D - A) = I - D^{-1}A$ has diagonal entries =1

All e-vals are in the circle

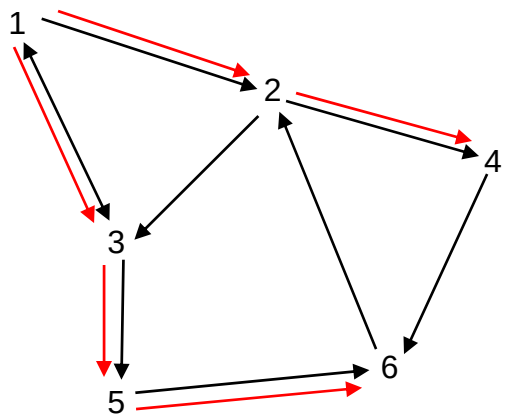


L has row sum = 0 implies there is an eigenvalue at s=0

$$0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$$



E-vals of -L



Theorem. Graph contains a spanning tree iff e-val at 0 is simple.

Then $\lambda_2 > 0$

Then (-L) has one e-val at zero and all the rest stable

Graph strongly connected implies exists a spanning tree

Consensus: Final Consensus Value

Closed-loop system with local voting protocol

$$\dot{x} = -Lx$$

Modal decomposition $x(t) = e^{-Lt}x(0) = \sum_{j=1}^{N} w_i^T e^{-\lambda_i t} v_i x(0) = \sum_{j=1}^{N} \left(w_i^T x(0) \right) e^{-\lambda_i t} v_i$

Let $\lambda_1 = 0$ be simple. Then at steady-state

$$x(t) \to v_1 e^{-\lambda_1 t} w_1^T x(0) = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} \gamma_1 & \cdots & \gamma_N \end{bmatrix} \begin{bmatrix} x_1(0) \\ \vdots \\ x_N(0) \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \sum_{j=1}^{N} \gamma_j x_j(0)$$

with $w_1^T = \begin{bmatrix} \gamma_1 & \cdots & \gamma_N \end{bmatrix}$ the normalized left e-vector of $\lambda_1 = 0$

Therefore $x_i(t) \to \sum_{j=1}^{N} \gamma_j x_j(0)$ for all nodes CONSENSUS

Consensus value depends on communication graph structure

Importance of left e-vector of $\lambda_1 = 0$

Convergence Rate

Closed-loop system with local voting protocol

$$\dot{x} = -Lx$$

Modal decomposition $x(t) = e^{-Lt}x(0) = \sum_{j=1}^{N} w_i^T e^{-\lambda_i t} v_i x(0) = \sum_{j=1}^{N} \left(w_i^T x(0) \right) e^{-\lambda_i t} v_i$

Let $\lambda_1 = 0$ be simple. Then for large t

$$x(t) \to v_2 e^{-\lambda_2 t} w_2^T x(0) + v_1 e^{-\lambda_1 t} w_1^T x(0) = v_2 e^{-\lambda_2 t} w_2^T x(0) + \underline{1} \sum_{j=1}^{N} \gamma_j x_j(0)$$

λ_2 determines the rate of convergence and is called the FIEDLER e-value

There is a big push to find expressions for the left e-vector for $\lambda_1 = 0$ and the Fiedler e-val λ_2

Olshevsky and Tsitsiklis

Fiedler e-val depends on left e-vect for $\lambda_1 = 0$

$$w_1^T = \begin{bmatrix} \gamma_1 & \cdots & \gamma_N \end{bmatrix}$$

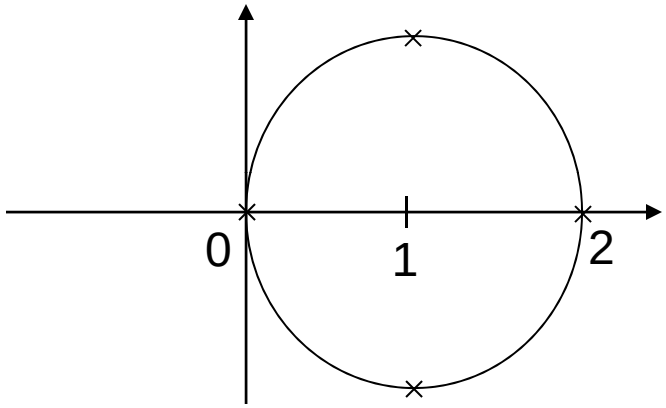
For a certain type of graph

$$\lambda_2 = 1 - \tfrac{1}{2} \min_{x \in S} \sum_{i=1}^{N} \sum_{j=1}^{N} \gamma_i a_{ij} (x_i - x_j)^2$$

E.G. if graph is k-cyclic (all circular paths have a GCD of k)

then e-vals are uniformly spaced around this circle

e.g. k=4



Then consensus time plot oscillates until steady-state

Consensus value is

$$x_i(t) \to \sum_{j=1}^{N} \gamma_j x_j(0)$$ = weighted average of initial conditions of nodes

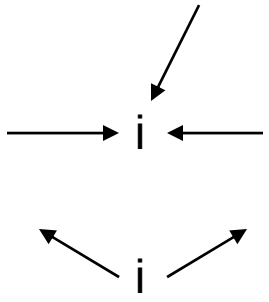
Consensus value depends on communication graph structure

A graph is balanced if in-degree=out-degree

$$A = [a_{ij}]$$

$$d_i = \sum_{j=1}^{N} a_{ij}$$ Row sum= in-degree

$$d_i^{\,o} = \sum_{i=1}^{N} a_{ij}$$ Column sum= out-degree



Balanced means that row sum= column sum

Then L has row sum=0 and column sum=0

$w_1^T L = 0$ means that left e-vector is also $w_1 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \underline{1}$

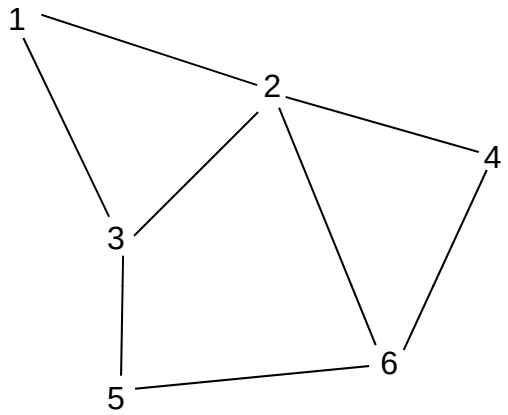
Then Consensus value is

$$x_i(t) \to \frac{1}{N} \sum_{j=1}^{N} x_j(0)$$ = average of initial conditions of nodes

Independent of graph structure

Undirected Graphs

e_{ij} is an edge if e_{ji} is an edge



$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

A is symmetric

Row sum= column sum so that in-degree= out-degree

L=D-A is symmetric and positive semidefinite

A symmetric graph is balanced

Connected undirected graph has average consensus value

Reynolds, Computer Graphics 1987

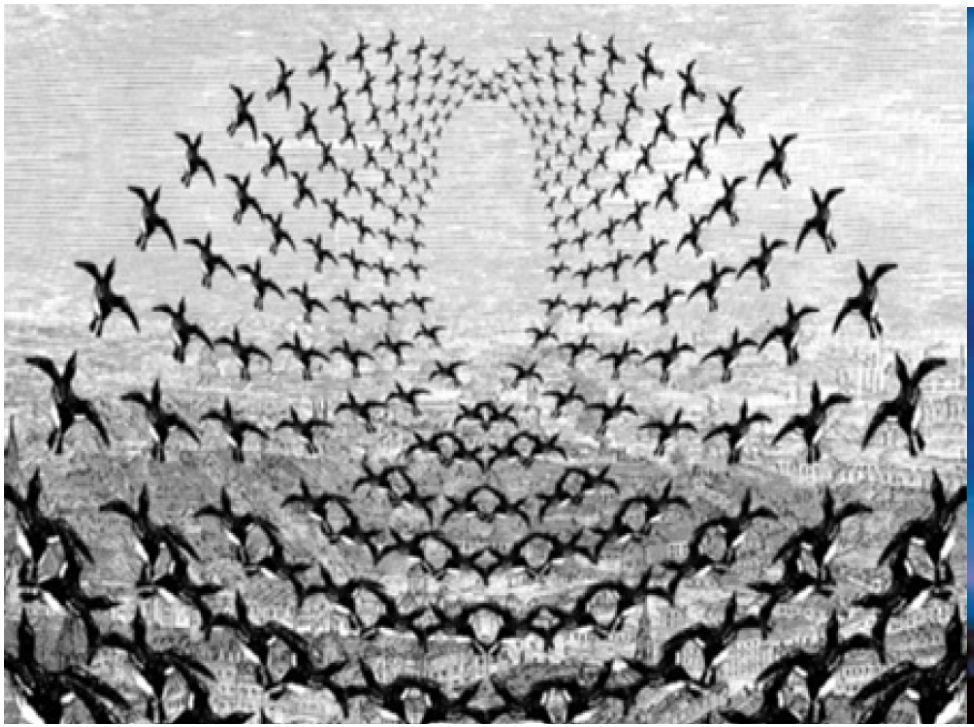
Flocking

Reynolds' Rules:

Alignment : align headings $\dot{\theta}_i = \sum_{j \in N_i} a_{ij}(\theta_j - \theta_i)$

Cohesion : steer towards average position of neighbors- towards c.g.

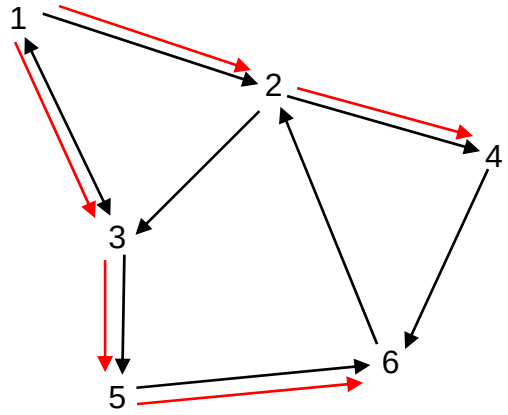
Separation : steer to maintain separation from neighbors







Consensus Leaders



Spanning tree
Spanning node

CONSENSUS $x_i(t) \to \sum_{j=1}^{N} \gamma_j x_j(0)$ for all nodes

with $w_1^T = [\gamma_1 \quad \cdots \quad \gamma_N]$ the normalized left e-vector of $\lambda_1 = 0$

Theorem. $\gamma_i \neq 0$ iff node i is a spanning node

So the consensus value is a weighted average of the initial conditions of all the spanning nodes

Corollary. Let the graph be a tree. Then all nodes converge to IC of the root node

Bounds for Fiedler e-val: Convergence Rate

Undirected Graphs Row sum = col sum, in-degree = out-degree, balanced

$$\lambda_2 \geq \frac{1}{D\, vol(G)}$$

D= diameter= length of longest path between 2 nodes

$$vol(G) = \sum_i d_i$$

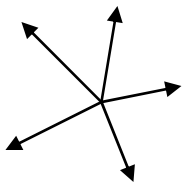
Strogatz Small World Networks are faster

$$\lambda_2 \leq \frac{n}{n-1}\delta, \text{ where } \delta \text{ is the minimum in-degree}$$

In-degree = out-degree

Directed Graphs- less is known

Star network has largest Fiedler e-val of any graph with the same number of nodes and edges



$$a_1(L) \leq \frac{n}{n-1}\min\{\text{max out degree, max in degree}\}$$

Book by Chai Wah Wu
Work of Fan Chung

OUT degree is important

Hovareshthi & Baras

Left e-vector for $\lambda_1 = 0$

$$w_1^T = [\gamma_1 \quad \cdots \quad \gamma_N]$$

For a connected undirected graph $\gamma_i = \dfrac{d_i + 1}{2L + N}$

where d_i is the out-degree= in-degree, N the number of nodes, L the number of edges

Olshevsky and Tsitsiklis

Bidirectional graph $a_{ij} \neq 0 \Rightarrow a_{ji} \neq 0$

Equal neighbor $a_{ij} = \frac{1}{n_i}, \quad a_{ii} = \frac{1}{n_i}$

$n_i = \left|N_i\right| + 1$ Number of neighbors +1 (i.e. include node itself)

Then left evector for $\lambda_1 = 0$

$$w_1^T = \begin{bmatrix} \gamma_1 & \cdots & \gamma_N \end{bmatrix}$$

$\gamma_i = \frac{n_i}{Vol(G)}$ where $Vol(G) = \sum_i n_i$

Consensus value is $x_i(t) \to \sum_{i=1}^{N} \gamma_i x_i(0) = \sum_{i=1}^{N} \frac{n_i}{Vol(G)} x_i(0)$

Run Two Consensus Algorithms at each node

Olshevsky & Tsitsiklis

Get rid of left-eigenvector dependence

State 1

$$\dot{y}_i = u_i, \quad , \qquad \boxed{y_i(0) = 1/n_i,} \qquad u_i = \sum_{j \in N_i} a_{ij}(y_j - y_i)$$

$$y_i(t) \to \sum_{i=1}^{N} \gamma_i y_i(0) = \sum_{i=1}^{N} \frac{n_i}{Vol(G)} \frac{1}{n_i} = \frac{N}{Vol(G)}$$

Learns global graph properties

State 2

$$\dot{z}_i = w_i \ , \qquad \boxed{z_i(0) = \frac{x_i(0)}{n_i},} \qquad w_i = \sum_{j \in N_i} a_{ij}(z_j - z_i)$$

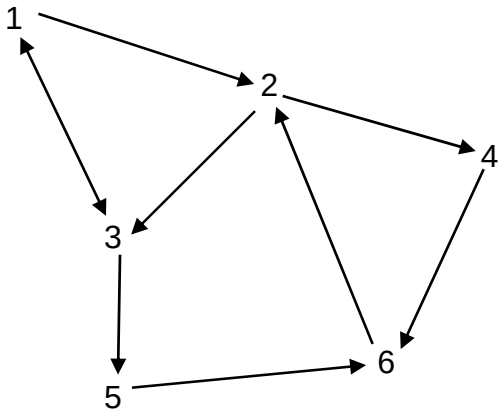
$$z_i(t) \to \sum_{i=1}^{N} \gamma_i z_i(0) = \sum_{i=1}^{N} \frac{n_i}{Vol(G)} \frac{x_i(0)}{n_i} = \frac{1}{Vol(G)} \sum_{i=1}^{N} x_i(0)$$

Set $x_i(t) = z_i(t) / y_i(t)$

then $x_i(t) \to \frac{1}{N} \sum_{i=1}^{N} x_i(0)$ Average consensus!

Independent of graph structure

Consensus Control for Swarm Motions

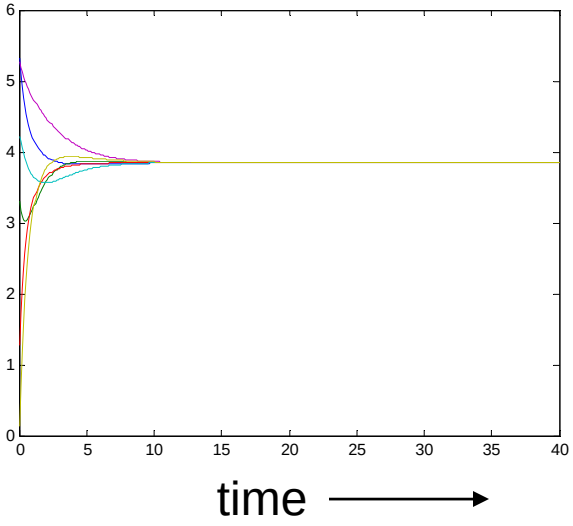


$$\dot{\theta}_i = \sum_{j \in N_i^c} a_{ij}(\theta_j - \theta_i)$$

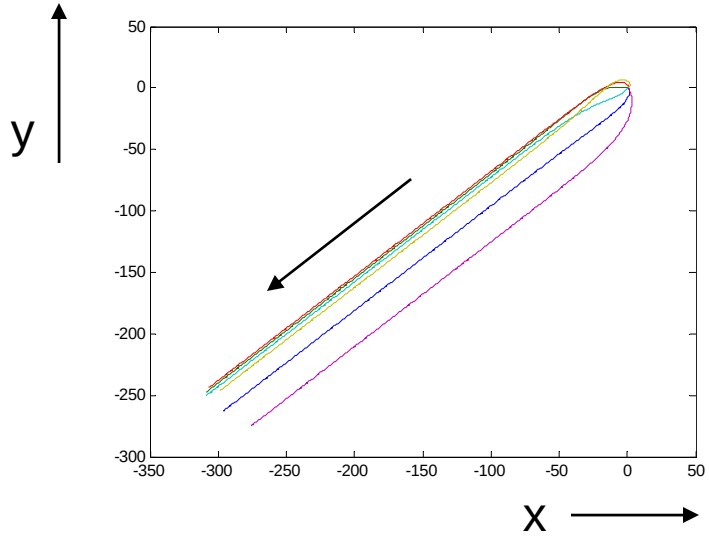
heading angle

$$\dot{x}_i = V\cos\theta_i$$

$$\dot{y}_i = V\sin\theta_i$$



Convergence of headings



Nodes converge to consensus heading

Trust Propagation and Consensus – Network Security

Inspired by social behavior in flocks, herds, teams

Foundation work by John Baras

Define ξ_{ij} as the trust that node i has for node j

$\xi_{ij} \in [-1,1]$

-1 ..... 0 1

Distrust no opinion complete trust

Define trust vector of node i as

$$\xi_i = \begin{bmatrix} \xi_{i1} \\ \xi_{i2} \\ \xi_{i3} \\ \xi_{i4} \\ \xi_{i5} \\ \xi_{i6} \end{bmatrix}$$

← Trust node i has for node 3

N vector

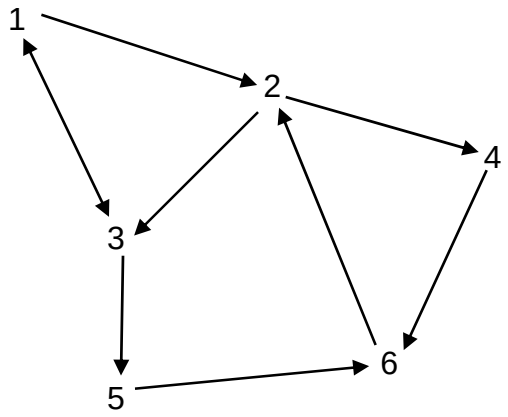
$$\xi \equiv \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_N \end{bmatrix} \in R^{N^2}$$

Standard local voting protocol $u_i = \sum_{j \in N_i} a_{ij}(\xi_j - \xi_i)$

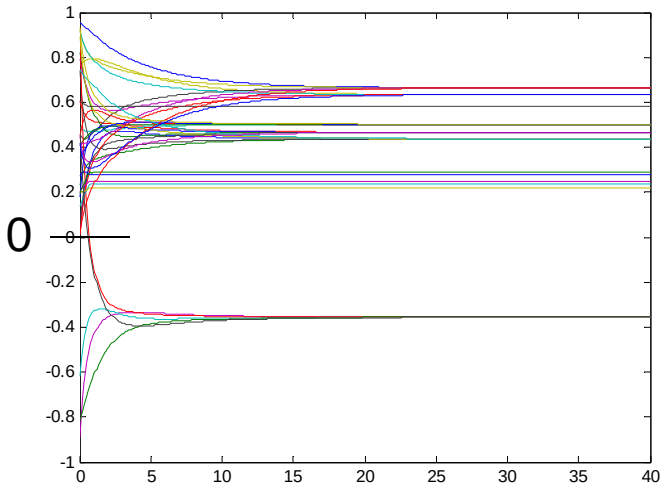
Difference of opinion with neighbors

Closed-loop trust dynamics $\dot{\xi} = -(L \otimes I_N)\xi$

Trust Propagation & Consensus



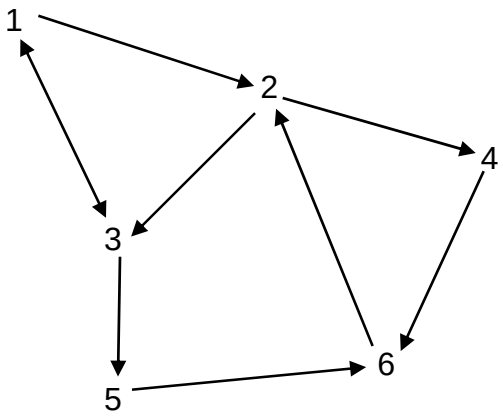
Nodes 1, 2, 4 initially distrust node 5



Other nodes agree that node 5 has negative trust

Convergence of trust

Trust-Based Control: Swarms/Formations

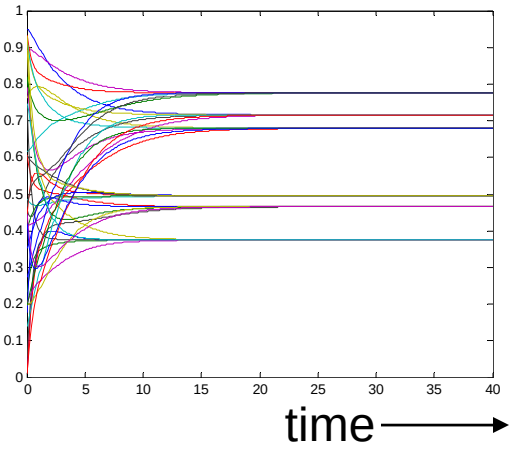


Trust dynamics $\dot{\xi}_i = \sum_{j \in N_i} a_{ij}(\xi_j - \xi_i)$

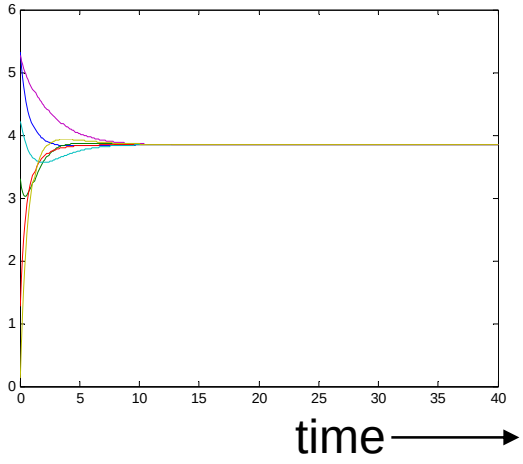
Motion dynamics $\dot{\theta}_i = \sum_{j \in N_i^c} \xi_{ij} a_{ij}(\theta_j - \theta_i)$ heading angle

$\dot{x}_i = V \cos\theta_i$

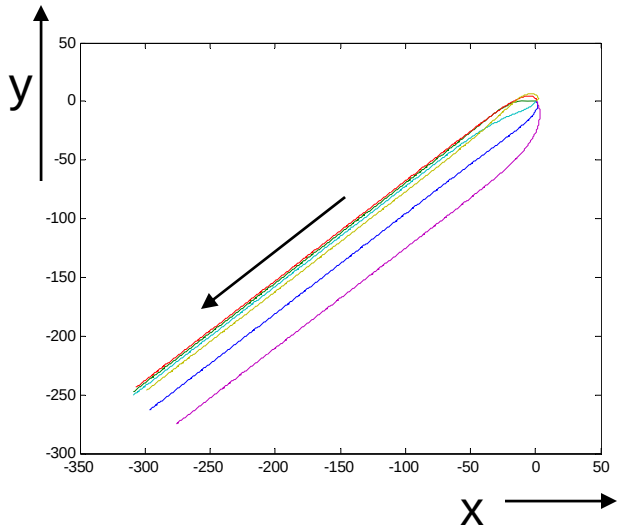
$\dot{y}_i = V \sin\theta_i$



Convergence of trust



Convergence of headings

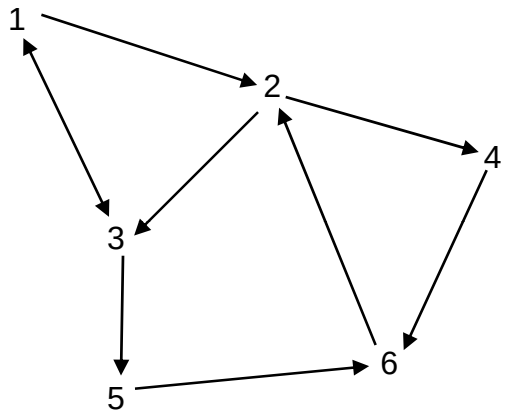


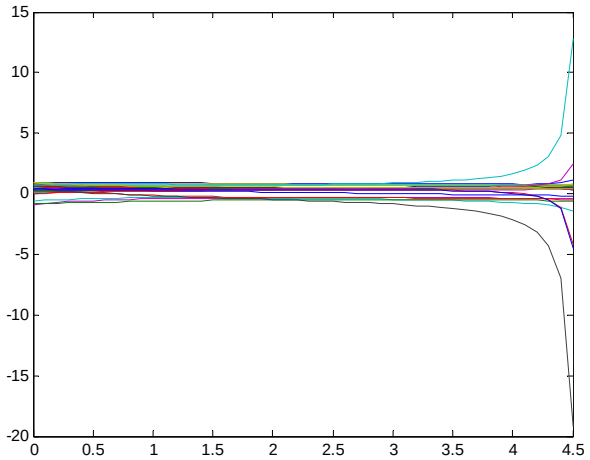
Nodes converge to consensus heading

Trust-Based Control: Swarms/Formations

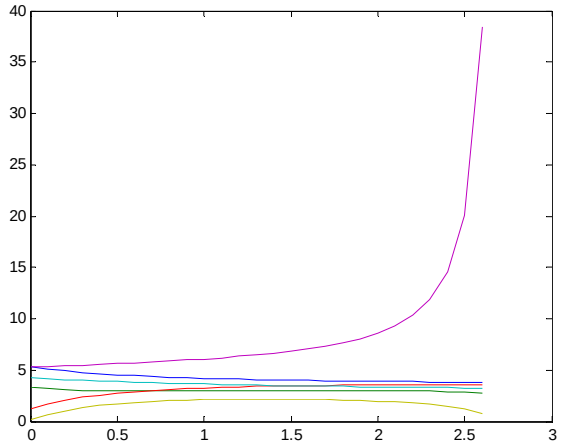
Malicious Node

$$\dot{\theta}_i = \sum_{j \in N_i^c} \xi_{ij} a_{ij} (\theta_j - \theta_i)$$





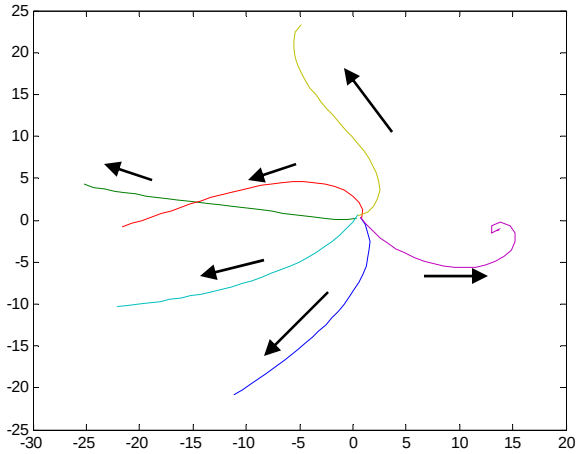
Divergence of trust



Divergence of headings

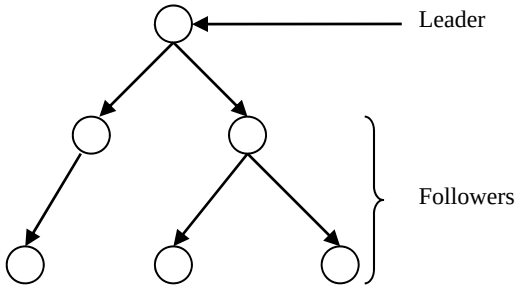
Node 5 injects
negative trust values

Internal attack
Malicious node puts out bad trust values
i.e. false *information*
c.f. virus propagation



Causes Unstable Formation

Consensus Control for Formations

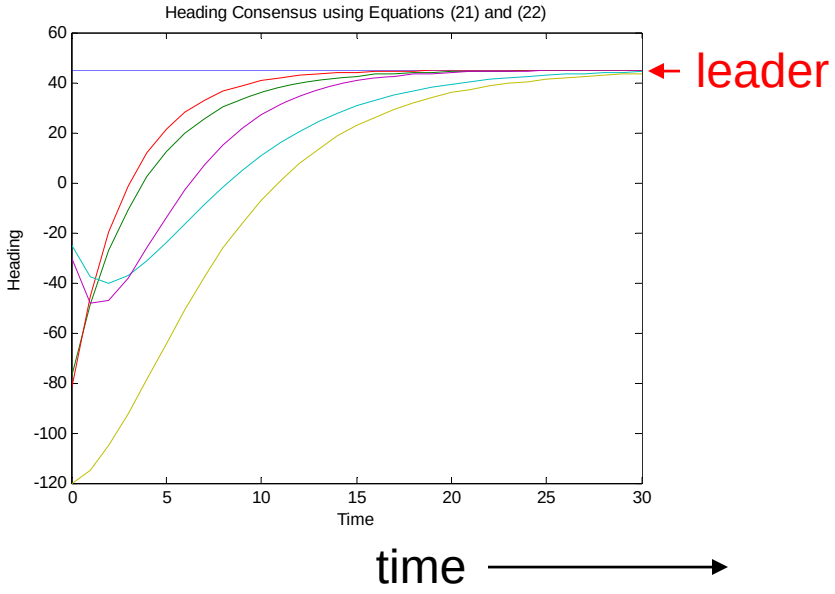


Formation- a Tree network

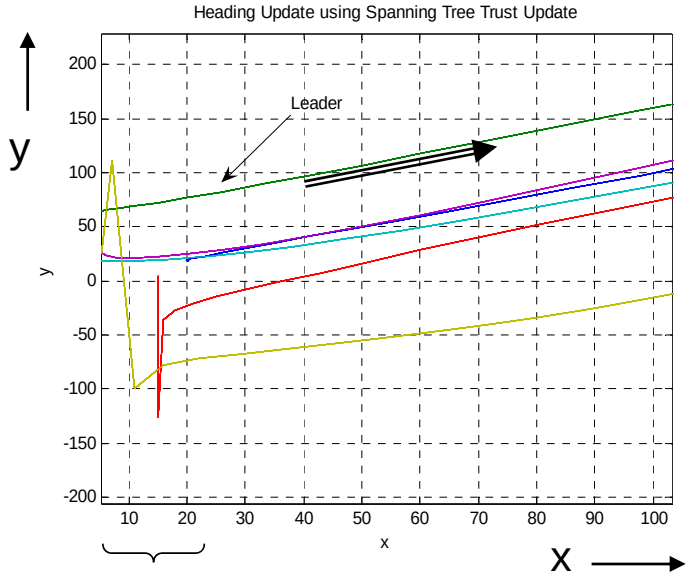
$$\dot{\theta}_i = \sum_{j \in N_i^c} a_{ij}(\theta_j - \theta_i)$$ heading angle

$$\dot{x}_i = V\cos\theta_i$$

$$\dot{y}_i = V\sin\theta_i$$



Convergence of headings

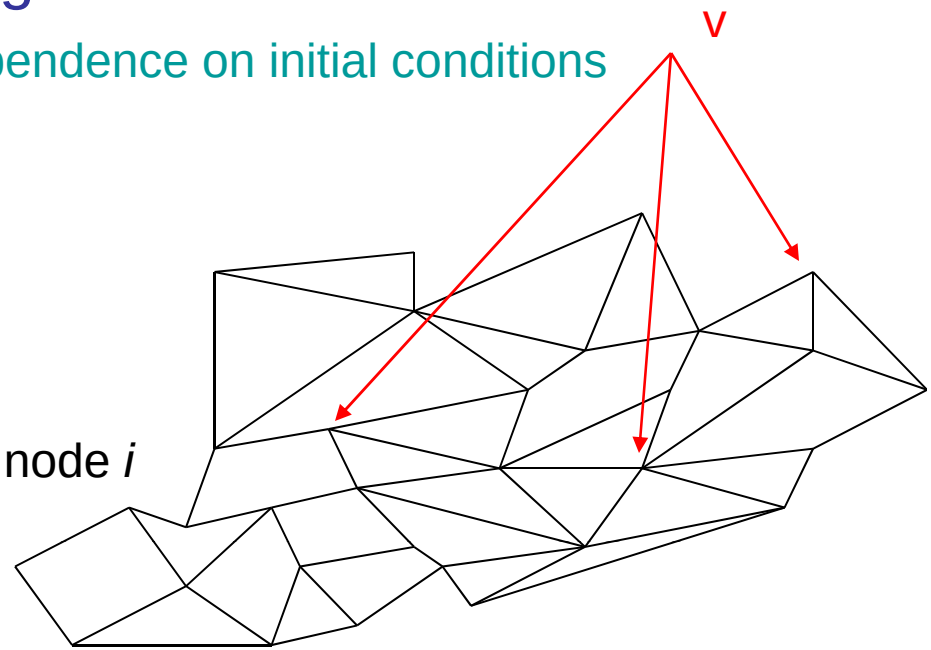


Nodes converge to heading of leader

Controlled Consensus: Pinning Control

Get rid of dependence on initial conditions

Node state $\dot{x}_i = u_i$



Local voting protocol with control node

$$u_i = \sum_{j \in N_i} a_{ij}(x_j - x_i) \quad + b_i(v - x_i)$$

$b_i \neq 0$ If control v is in the neighborhood of node *i*

$$u_i = -\sum_{j \in \bar{N}_i} a_{ij} x_i + \sum_{j \in N_i} a_{ij} x_j \quad + b_i v$$

Control node is in some neighborhoods $\bar{N}_i$

$$\dot{x} = -(L + B)x + B\underline{1}v \qquad B = diag\{b_i\}$$

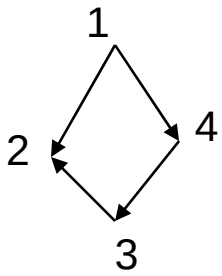
Theorem. Let at least one $b_i \neq 0$. Then L+B is nonsingular with all e-vals positive and -(L+B) is asymptotically stable

So initial conditions of nodes in graph A go away.
Consensus value depends only on v

In fact, v is now the only spanning node

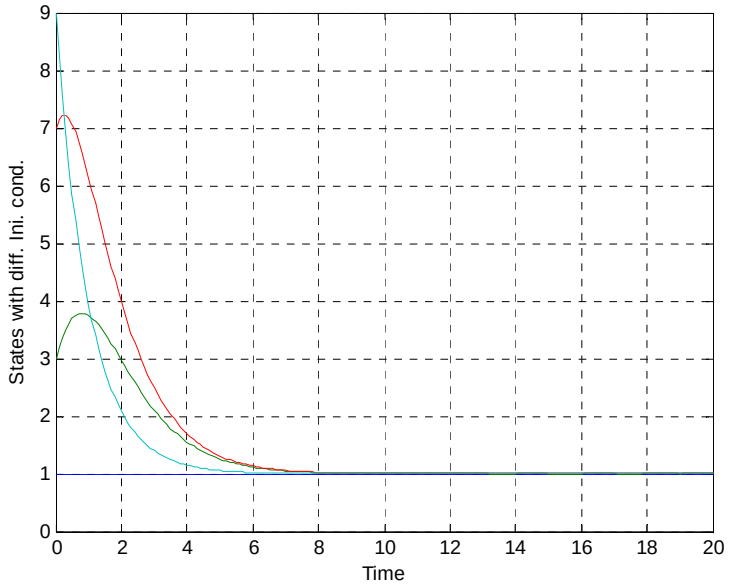
Controlled Consensus

Original network



A = [0 0 0 0;
1 0 1 0;
0 0 0 1;
1 0 0 0];

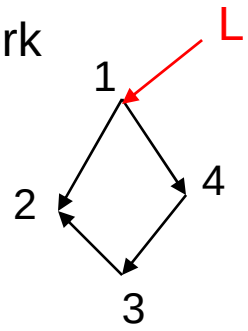
Lamda's = 0 1 1 2



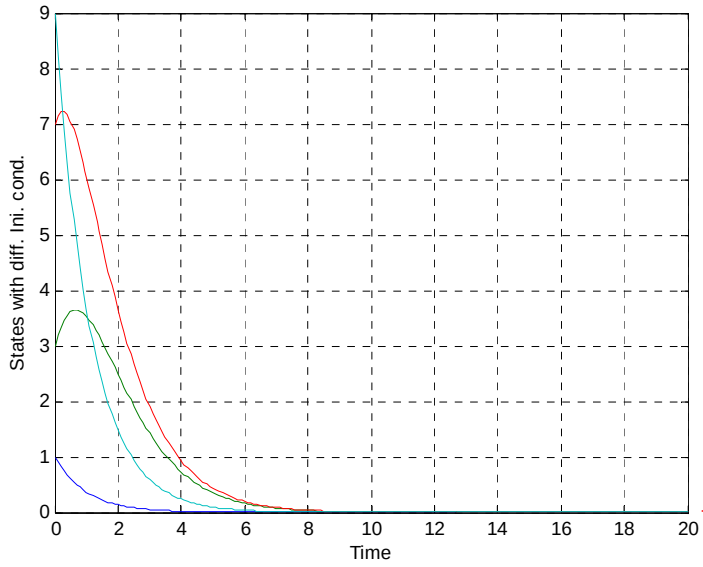
Average of ICS

Consensus time approx 7.5 sec

Controlled network



Lamda's = 0 1 1 1 2



Leader's IC

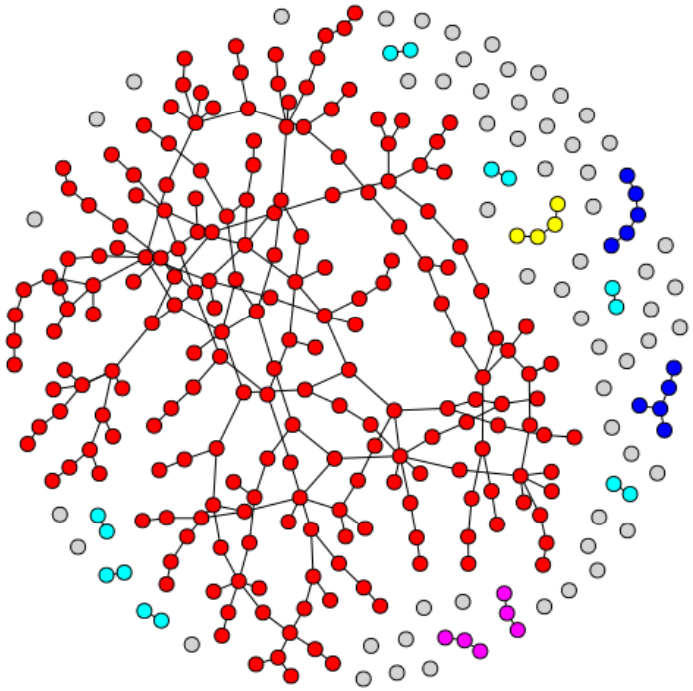
Consensus time approx 8 sec

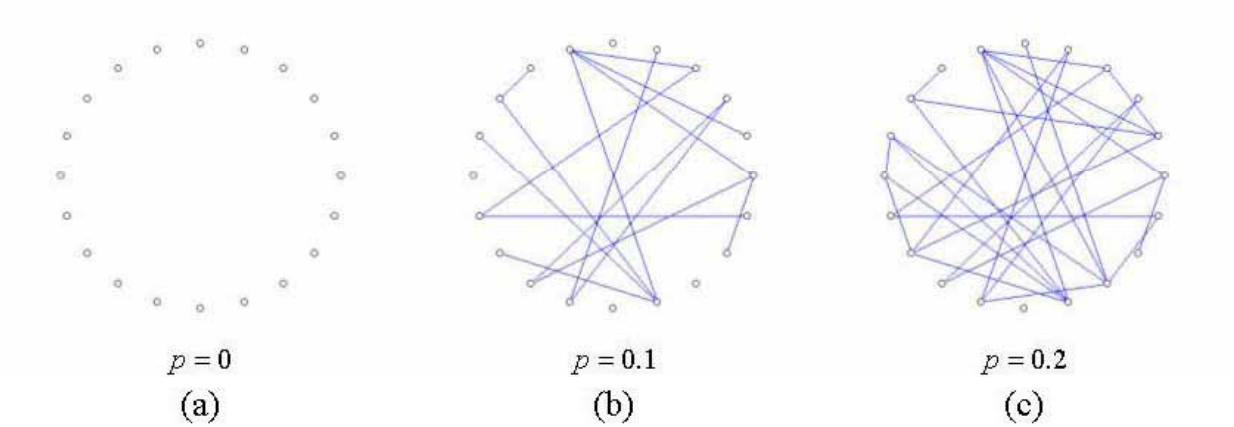
Random Graphs – Erdos and Renyi

N nodes
Two nodes are connected with probability p independent of other edges

m= number of edges

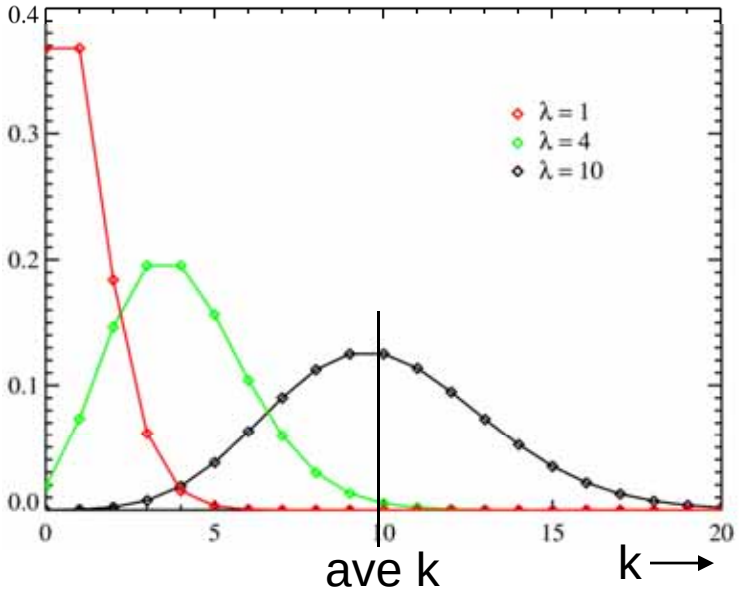
There is a critical threshold $m_0(n) = N/2$ above which a large connected component appears – giant clusters





J.J. Finnigan, Complex science for a complex world

Connectivity- degree distribution is Poisson
Homogeneity- all nodes have about the same degree



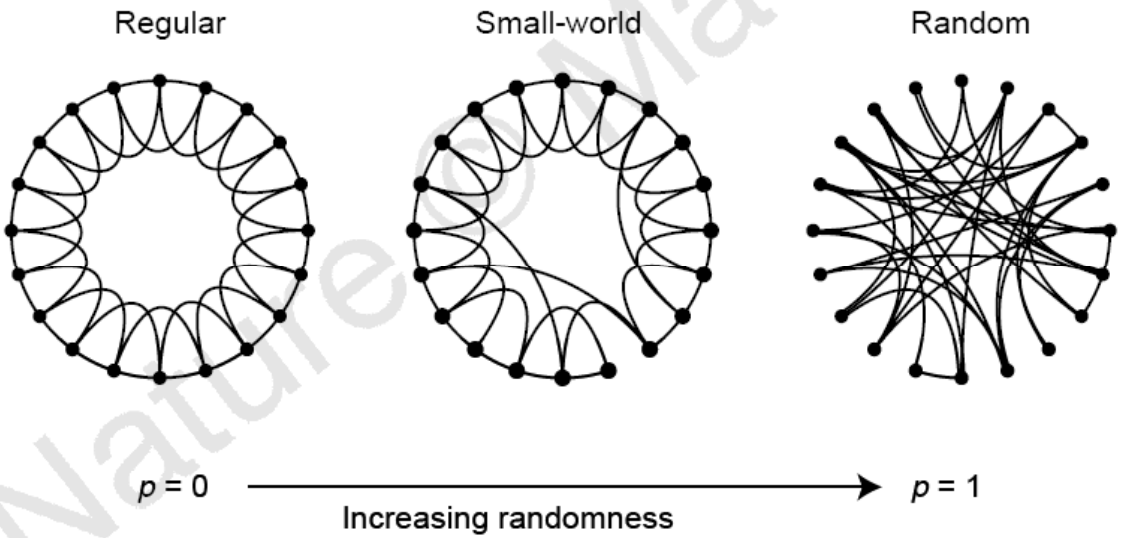
Poisson degree distribution
most nodes have about the same degree
ave(k) depends on number of nodes

Small World Networks- Watts and Strogatz

Start with a regular lattice
With probability p, rewire an edge to a random node.

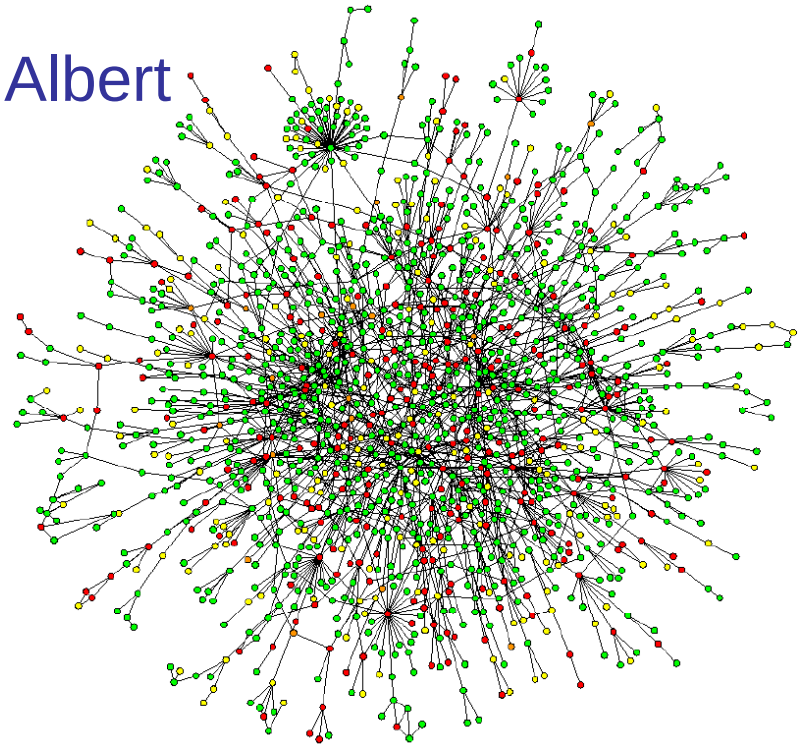
Connectivity- degree distribution is Poisson
Homogeneity – all nodes have about the same degree

Small diameter (longest path length)
Large clustering coeff.- i.e. neighbors are connected



Watts & Strogatz, Nature 1998

Scale-Free Networks– Barabasi and Albert



Start with m_0 nodes
Add one node at a time:
connect to m other nodes

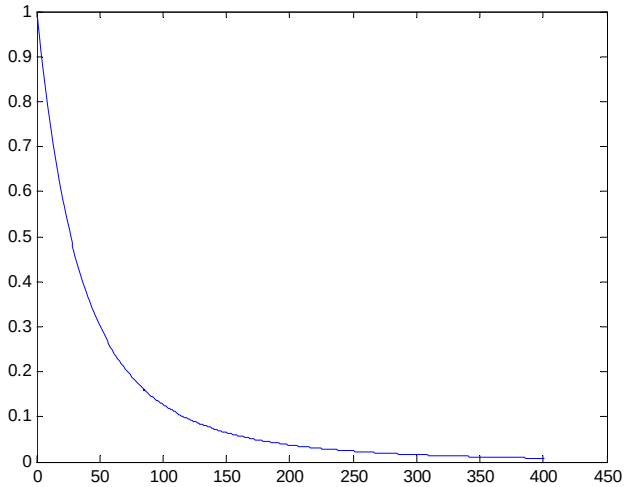
with probability $P(i) = \frac{d_i + 1}{\sum_j (d_j + 1)}$

i.e. with highest probability to biggest nodes
(rich get richer)

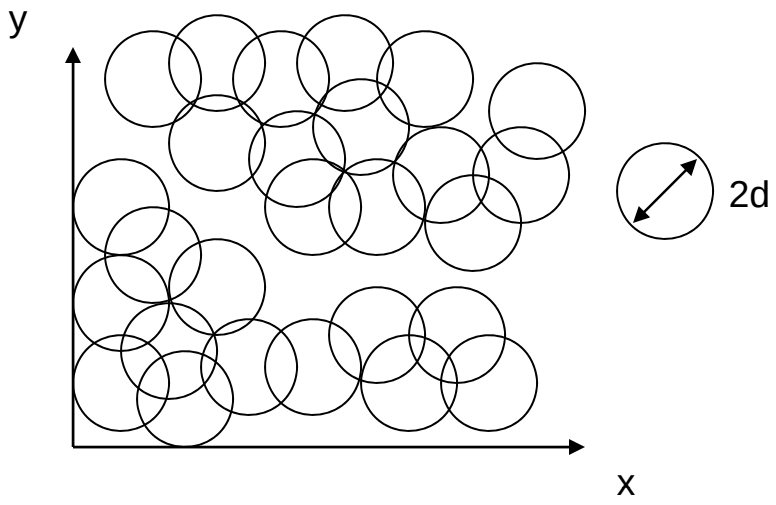
Nonhomogeneous- some nodes have large degree, most have small degree
Scale-Free- degree has power law degree distribution



$$P(k) = \frac{2m^2}{k^3}$$



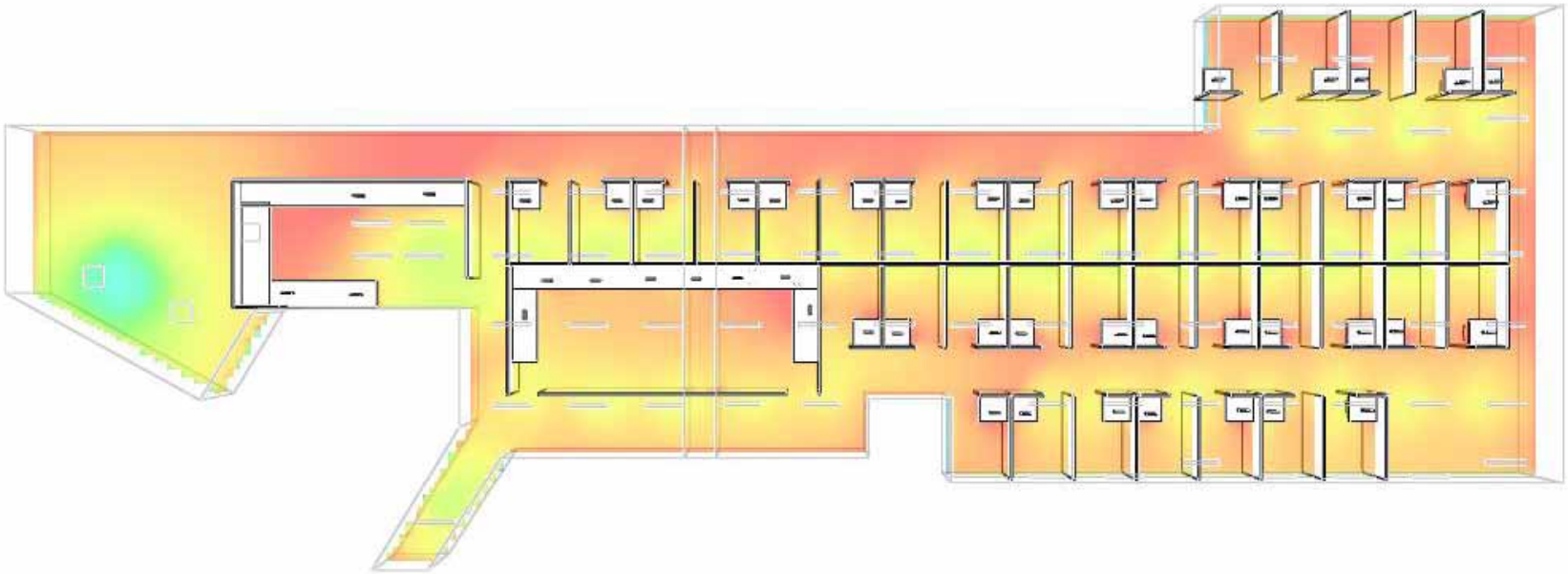
Proximity Graphs



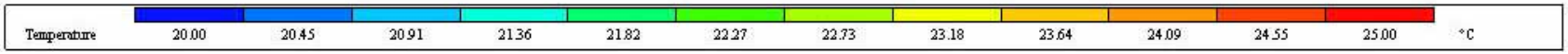
Randomly select N points in the plane
Draw an edge (i,j) if distance between nodes i and j is within d

When is the graph connected?
for what values of (N,d)
What is the degree distribution?

Balancing HVAC Ventilation Systems

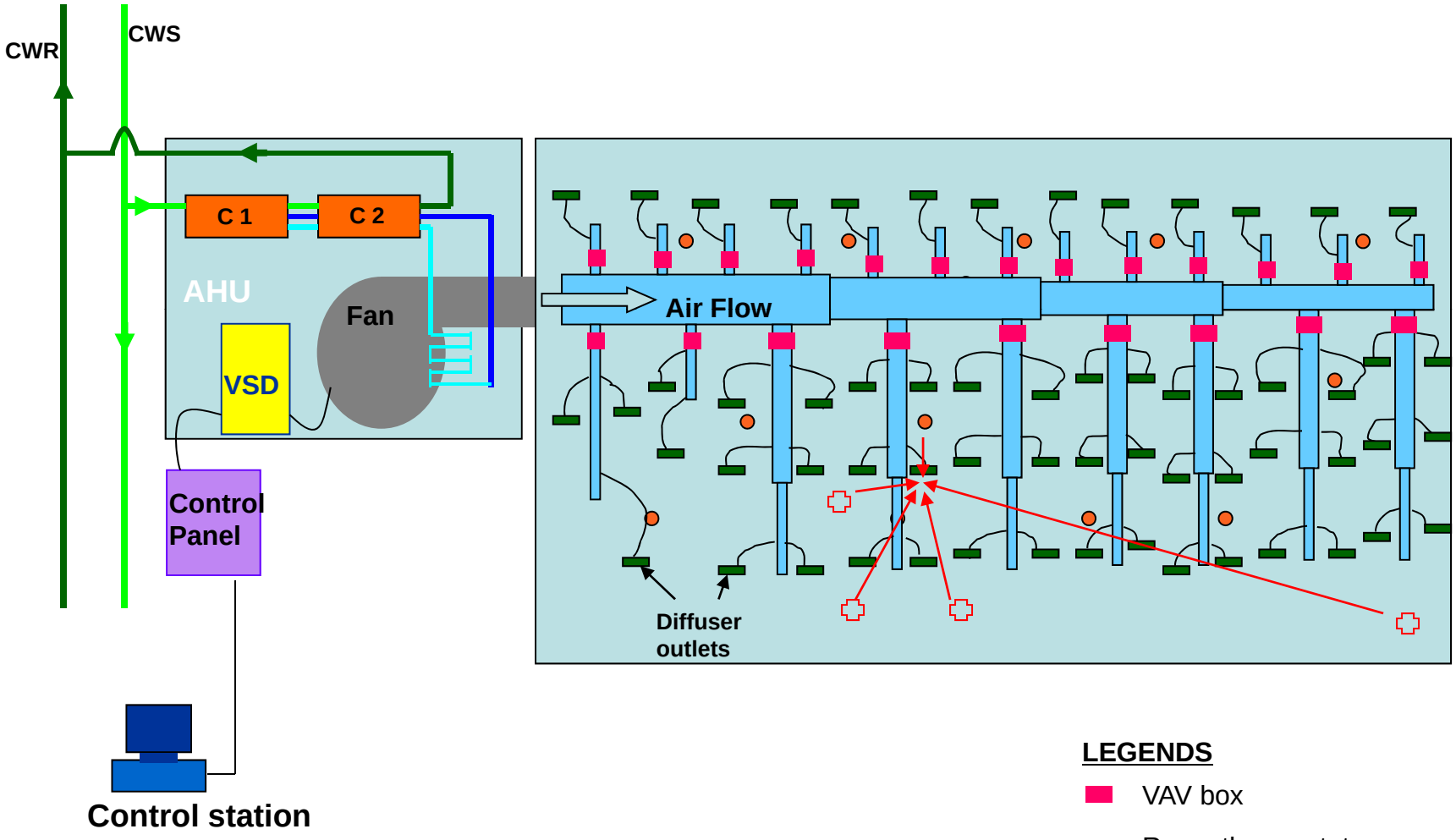


SIMTech 5th floor temperature distribution



Leong Wai Yie

Automated VAV control system



Leong Wai Yie

Adjust Dampers for desired Temperature distribution

Temperature dynamics

$$x_i(k+1) = x_i(k) + f_i(x) + u_i(k)$$

Unknown $f_i(x)$

Control damper position based on local voting protocol

$$u_i(k) = \frac{1}{n_i + 1}\gamma_i(k)\sum_{j \in N_i} a_{ij}(x_j(k) - x_i(k))$$

$$\gamma_i(k) = 1, \tfrac{1}{2}, \tfrac{1}{4}, \ldots$$

Under certain conditions this converges to steady-state desired temp. distribution

How about consensus control for Building Egress?

Second Order Consensus – Kevin Moore and Wei Ren

Consensus works because the closed-loop system is Type I.

$$\dot{x} = -Lx$$

has an integrator- simple e-val at 0.

Let each node have an associated state $\ddot{x}_i = u_i$

Second-order local voting protocol

$$u_i = \gamma_0 \sum_{j \in N_i} a_{ij}(x_j - x_i) + \gamma_1 \sum_{j \in N_i} a_{ij}(\dot{x}_j - \dot{x}_i)$$

Closed-loop system

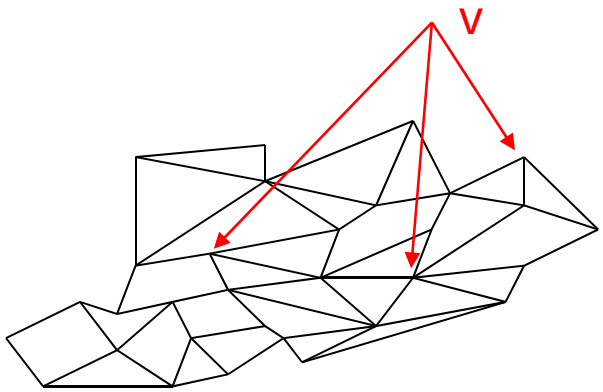
$$\dot{x} = \begin{bmatrix} 0 & I \\ -\gamma_0 L & -\gamma_1 L \end{bmatrix} x$$

Reaches consensus in both x_i, $\dot{x}_i$
iff graph has a spanning tree and gains are chosen for stability

Has 2 integrators- Can follow a ramp consensus input

Second Order Controlled Consensus for Position Offset Control – Kevin Moore and Wei Ren

node state $\ddot{x}_i = u_i$



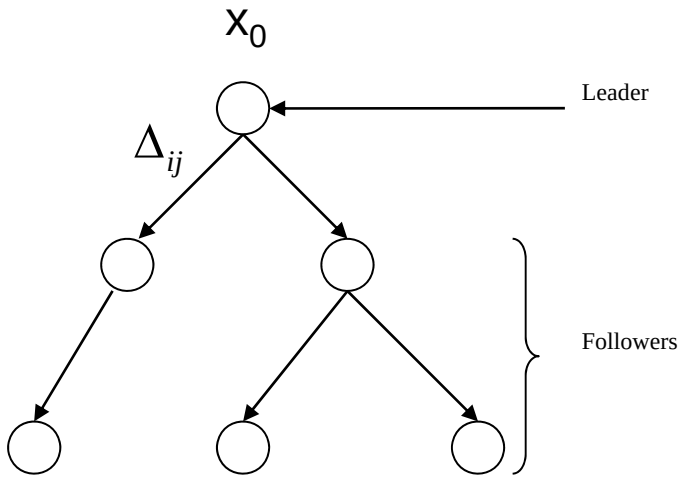
Second-order controlled protocol

$$u_i = \gamma_0 \sum_{j \in N_i} \left[a_{ij}(x_j - x_i) - \Delta_{ij} \right] + \gamma_1 \sum_{j \in N_i} a_{ij}(\dot{x}_j - \dot{x}_i) \quad + \; b_i \left[(x_0 - x_i) + (\dot{x}_0 - \dot{x}_i) \right]$$

where node 0 is a leader node.

Δ_{ij} is a desired separation vector

Good for formation offset position control

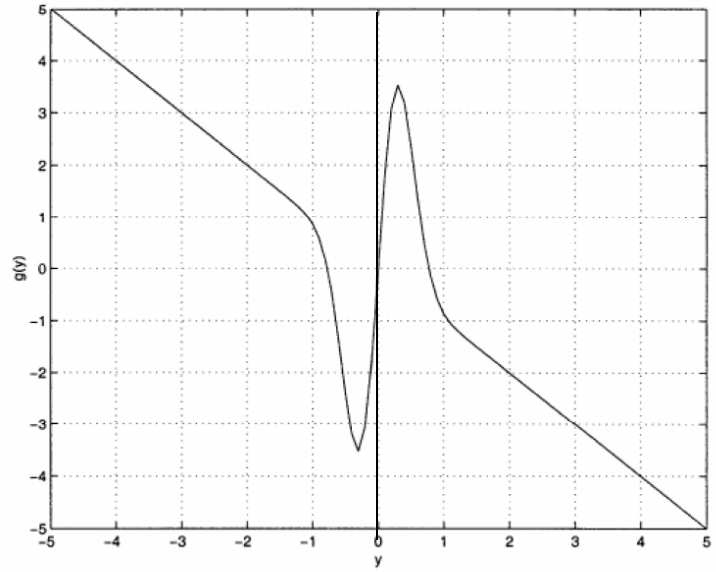


Swarm Stability Analysis

Gazi & Passino 2003, IEEE TAC

$$\dot{x}_i = \sum_{j \neq i} g(x_i - x_j)$$

$$g(y) = -y\left(a - b\exp\left(-\frac{\|y\|^2}{c}\right)\right)$$



c.g. motion is invariant
All agents converge to c.g.
form a hyperball of
constant radius and increasing density

Locust Swarm



Results of Gazi and Passino

$$g(y) = -y(a - b\exp\left(-\frac{\|y\|^2}{c}\right)$$

1. Center of gravity of swarm is stationary

$$\frac{d}{dt}\bar{x} = \frac{d}{dt}\frac{1}{N}\sum_{i=1}^{N} x_i(t) = 0$$

2. All states converge in finite time to the region $\|x_i - \bar{x}\| \le \frac{b(N-1)}{aN} < \frac{b}{a}$

   and the final density is $\rho > \frac{3a^3}{4\pi b^3} N$

3. Let nodes have finite body size of sphere with radius η

   Then all states converge to the region $\|x_i - \bar{x}\| \le \eta M^{1/3}$

   and the final density is $\rho > \frac{3}{4\pi\eta^3}$

Modeling Crowd Behavior in Stress Situations

Helbring, Farkas, Vicsek, Nature 2000

Consensus term

Interaction pot. field

Wall pot. field

$$m_i \frac{d\mathbf{v}_i}{dt} = m_i \frac{v_i^0(t)\mathbf{e}_i^0(t) - \mathbf{v}_i(t)}{\tau_i} + \sum_{j(\neq i)} \mathbf{f}_{ij} + \sum_W \mathbf{f}_{iW}$$

desired speed v_i^0 in a certain direction $\mathbf{e}_i^0$,

Repulsive force

Radial compression term

Tangential friction term

$$\mathbf{f}_{ij} = \left\{A_i \exp[(r_{ij} - d_{ij})/B_i] + kg(r_{ij} - d_{ij})\right\}\mathbf{n}_{ij} + \kappa g(r_{ij} - d_{ij})\Delta v_{ji}^t \mathbf{t}_{ij} \quad (2)$$

where the function $g(x)$ is zero if the pedestrians do not touch each other ($d_{ij} > r_{ij}$), and is otherwise equal to the argument x.

Herd and Panic Behavior During Emergency Building Egress

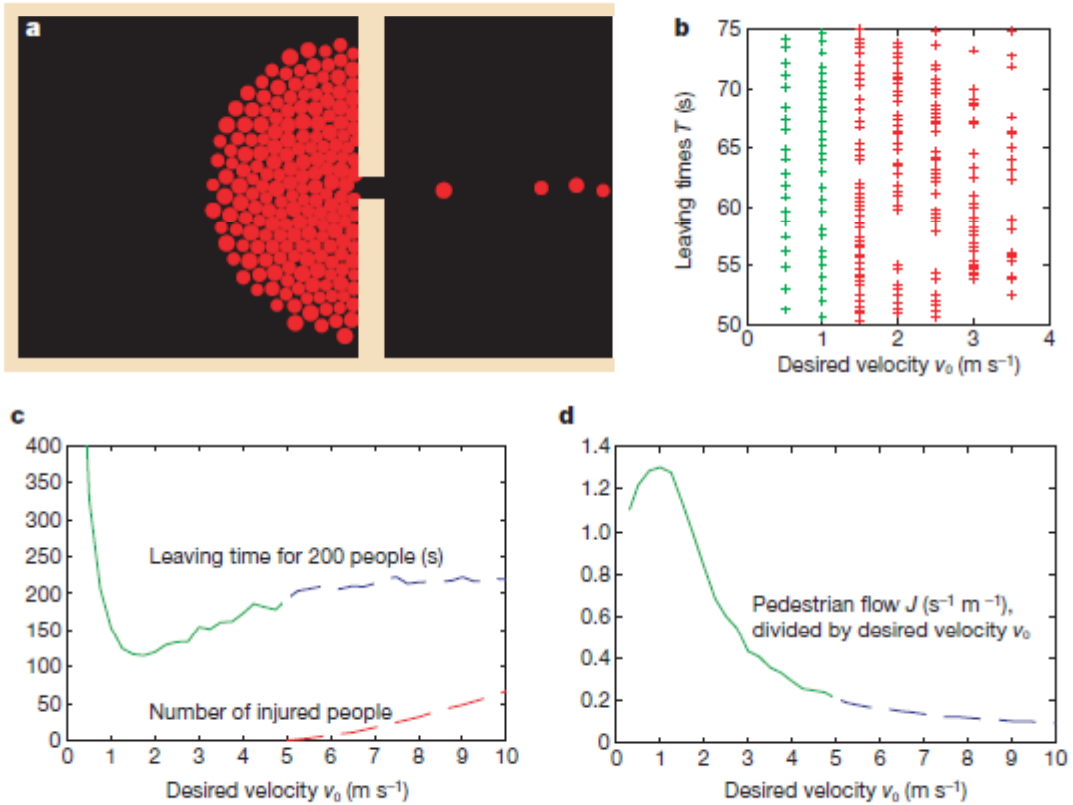


Figure 1 Simulation of pedestrians moving with identical desired velocity $v_i^0 = v_0$ towards the 1-m-wide exit of a room of size 15 m × 15 m. **a**, Snapshot of the simulation. Dynamic simulations are available at http://angel.elte.hu/~panic/. **b**, Leaving times of pedestrians for various desired velocities v_0. Irregular outflow due to clogging is observed for high desired velocities ($v_0 \geq 1.5\,\mathrm{m\,s^{-1}}$, red plus signs). **c**, Under conditions of normal walking, the time for 200 pedestrians to leave the room decreases with growing v_0. Desired velocities higher than $1.5\,\mathrm{m\,s^{-1}}$ reduce the efficiency of leaving, which becomes particularly clear when the outflow J is divided by the desired velocity (**d**). This is due to pushing, which causes additional friction effects. Moreover, above a desired velocity of about $v_0 = 5\,\mathrm{m\,s^{-1}}$ (corresponding to dashed lines in **c** and **d**) people are injured and become non-moving obstacles for others, if the sum of the magnitudes of the radial forces acting on them divided by their circumference exceeds a pressure of $1{,}600\,\mathrm{N\,m^{-1}}$ (ref. 5). Owing to the above 'faster-is-slower effect', panics can be triggered by pedestrian counterflows[2], which cause delays to the crowd intending to leave. This makes the stopped pedestrians impatient and pushy which may be described by increasing the desired velocity according to $v_i^0(t) = [1 - p_i(t)]v_i^0(0) + p_i(t)v_i^{\max}$, where $v_i^0(0)$ is the initial, and $v_i^{\max}$ the maximum desired velocity. The time-dependent parameter $p_i(t) = 1 - \bar{v}_i(t)/v_i^0$, where $\bar{v}_i(t)$ denotes the average speed in the desired direction of motion, is a measure of impatience. Altogether, long waiting times increase the desired velocity, which can produce inefficient outflow. This further increases the waiting times, and so on, so that this tragic feedback can eventually trigger panics. It is therefore imperative to have sufficiently wide exits and to prevent counterflows when big crowds want to leave.

Helbring, Farkas, Vicsek, Nature 2000

Reynolds, Computer Graphics 1987

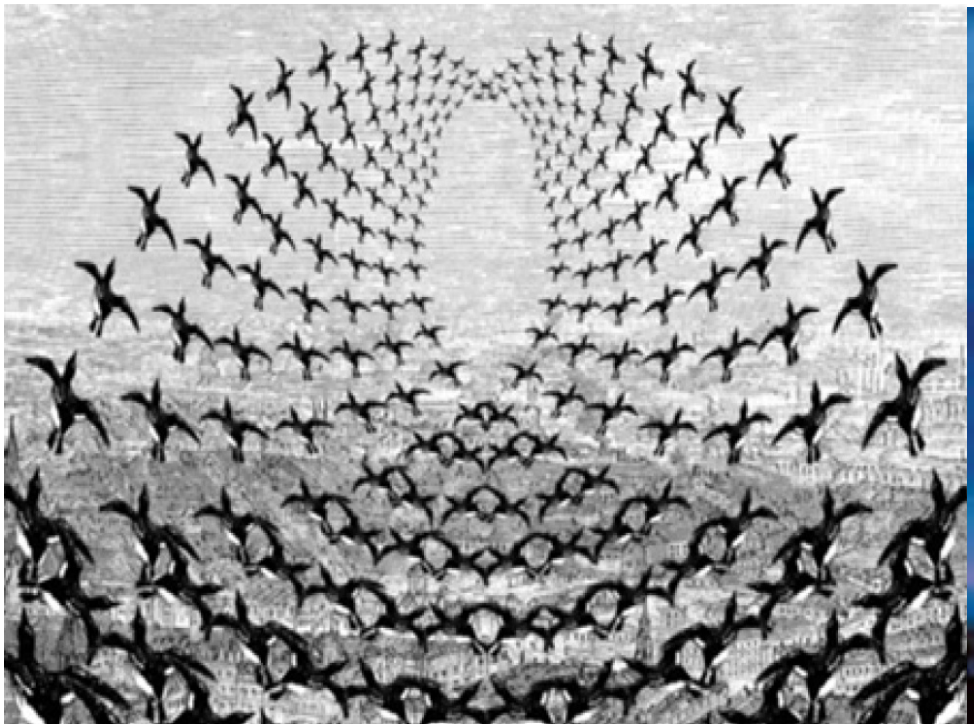
Flocking

Reynolds' Rules:

Alignment : align headings $\dot{\theta}_i = \sum_{j \in N_i} a_{ij}(\theta_j - \theta_i)$

Cohesion : steer towards average position of neighbors- towards c.g.

Separation : steer to maintain separation from neighbors





Synchronization

Spong and Chopra

$$\dot{x}_i = f_i(x_i) + g(x_i)u_i$$
$$y_i = h(x_i)$$

passive

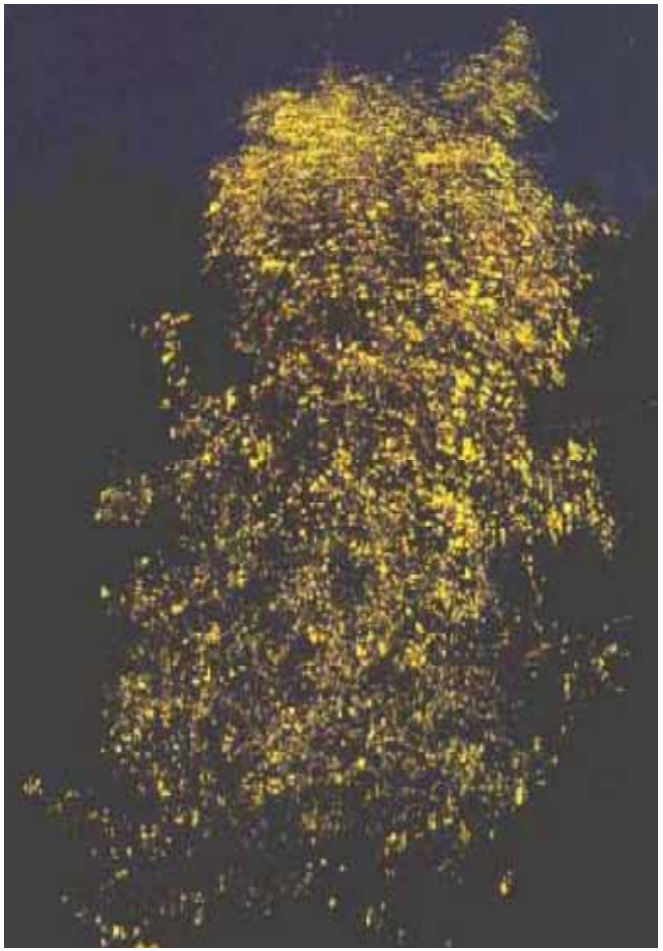
$$V_i(x_i) - V_i(x_i(0)) \le \int_0^t u_i^T(s) y_i(s)\, ds$$

Storage function

Synchronize if $y_i(t) \to y_j(t), \quad all\ i, j$

Local voting protocol

$$u_i = \sum_{j \in N_i} K(y_j - y_i)$$

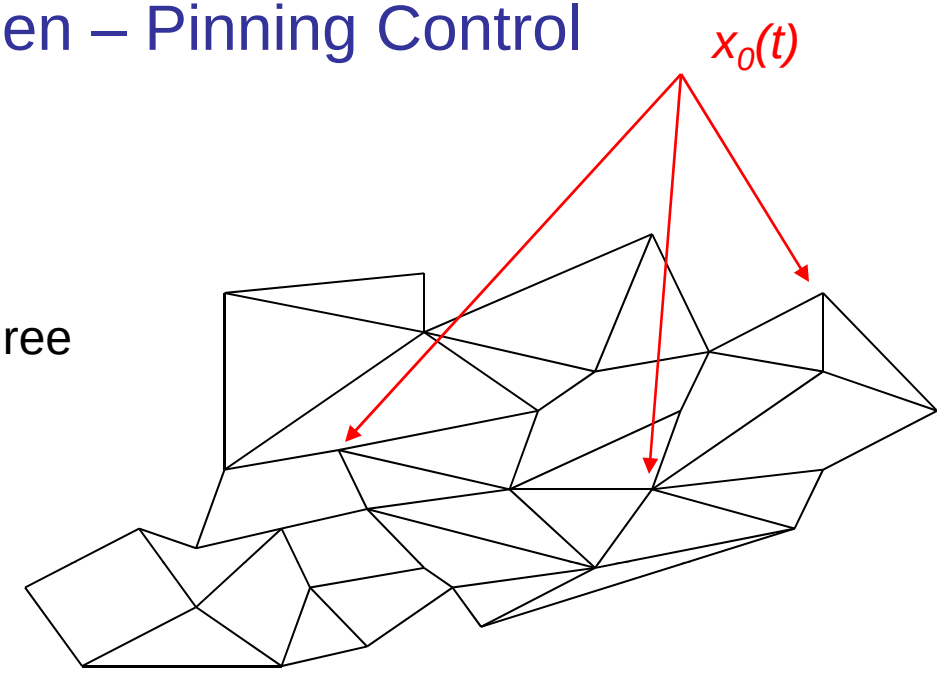


Fireflies synchronize

Result -
Let the communication graph be balanced. Then the agents synchronize.

Circadian rhythm

Synchronization : Ron Chen – Pinning Control



Connected undirected graphs

Leader node dynamics $\dot{x}_0 = f(x_0)$

$d_i = \sum_{j \neq i} a_{ij} = \sum_{j \neq i} a_{ji}$ In-degree = out-degree

$a_{ii} = -d_i$ Diffusivity condition

Pinning Control – inputs to some nodes

$$\dot{x}_i = f(x_i) + c\sum_j a_{ij} C(x_j - x_i) + cb_i C(x_0 - x_i)$$

$b_i = 1$ if node *i* is pinned

$$\dot{x} = \left(f(x) - c(L+B) \otimes Cx \right) + cB \otimes C\underline{1}x_0$$

$B = diag(b_i)$

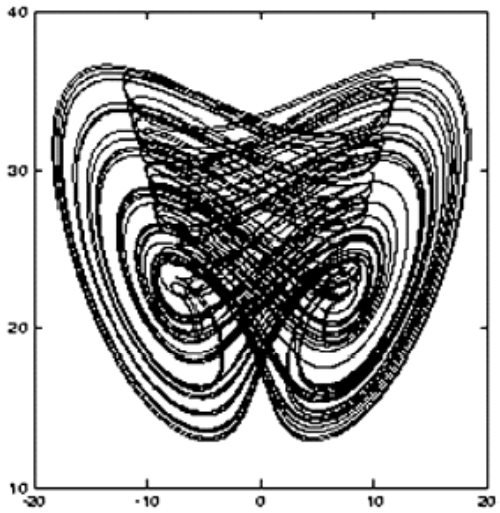
$L + B = D + B - A$ has e-vals λ_i

Results –
Node motions synchronize if $\left(\frac{\partial f(x)}{\partial x} - c\lambda_i C \right)$ is stable

Pin to the biggest node = highest degree node= highest social standing- c.f. Baras

Must have control gain c big enough

Synchronization of Chaotic node dynamics – Ron Chen



Chen's attractor
node dynamics

Pinning control of largest node
(for increasing coupling strengths)

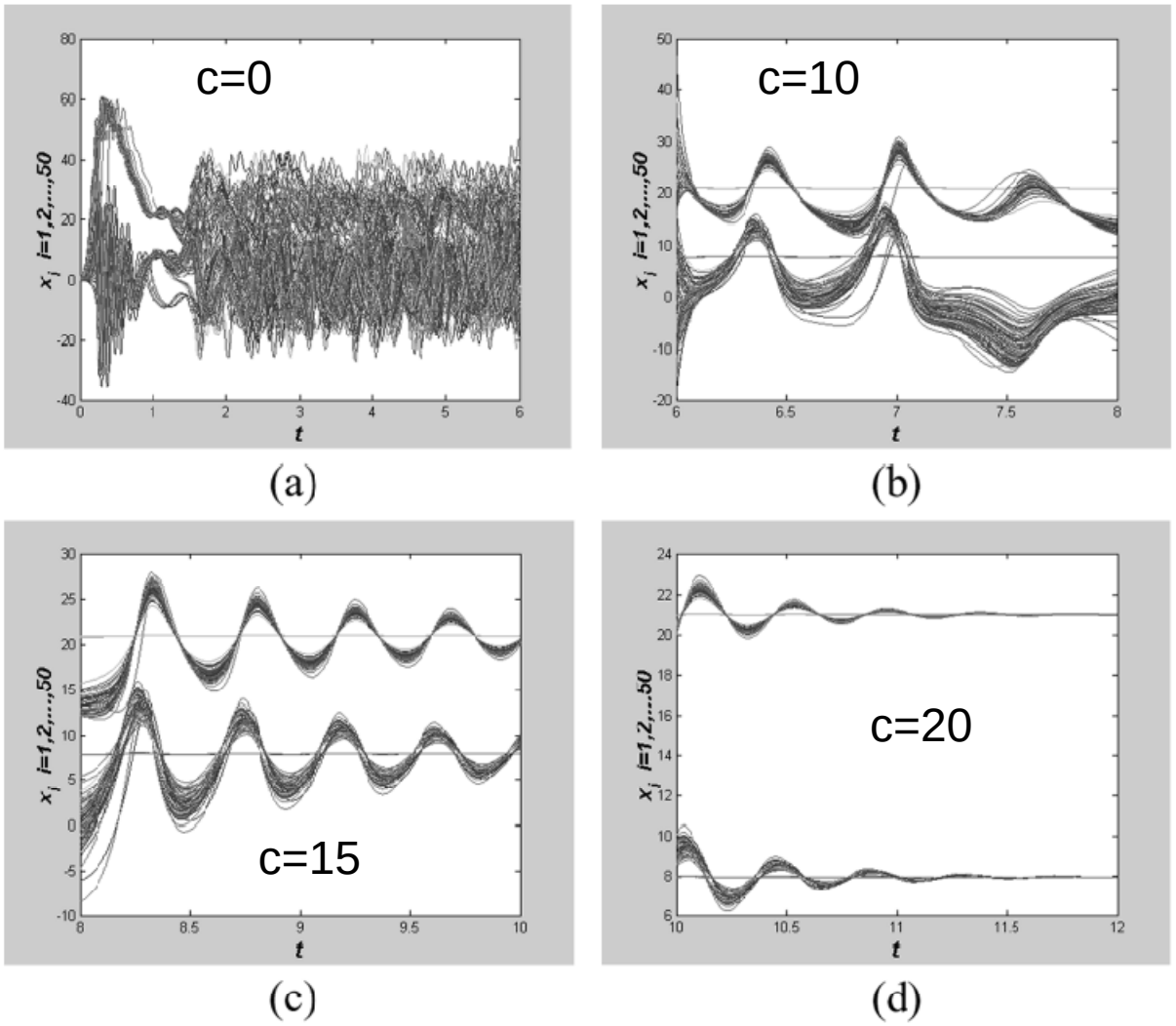


Fig. 8. Specifically pinning the biggest node of 19 degrees in a 50-node network generated by the B–A scale-free model: (a)–(d) are stabilizing phases with different coupling strengths. (a) $c = 0, d = 0$. (b) $c = 10, d = 1000$. (c) $c = 15, d = 1000$. (d) $c = 20, d = 1000$.


Thank you everyone!

Zhihua Qu 2009 book

$A \geq 0$ with row sum positive= d_i

$L = D - A$ is an M matrix

$$M = \begin{bmatrix} + & \leq 0 \\ \leq 0 & + \end{bmatrix}$$

Off-diagonal entries ≤ 0

Principal minors nonnegative – singular M matrix

Nonsingular M matrix if all principal minors positive

L also has all row sums = 0

Do not confuse with stochastic matrix

$E \geq 0$ is row stochastic if all row sums =1

$E = e^{-Lt}$ is row stochastic with positive diagonal elements

Let $E \geq 0$ be row stochastic. Then $A=I-E$ is an M matrix with row sums zero.

Discrete-time voting protocol gives stochastic c.l. matrix $x_{k+1} = Ex_k$

$$E = (I + D)^{-1}(I + A) = I - (I + D)^{-1} L$$

Irreducibility

Matrix *E* is reducible if it can be brought by row/column permutations to the form

$$\begin{bmatrix} * & 0 \\ * & * \end{bmatrix}$$

Two matrices that are similar using permutation matrices are said to be cogredient.

A graph *G(A)* is strongly connected iff its adjacency matrix A is irreducible.

A reducible matrix *E* can be brought by a permutation matrix *T* to the lower block triangular (LBT) Frobenius canonical form

$$F = TET^T = \begin{bmatrix} F_{11} & 0 & \cdots & 0 \\ F_{21} & F_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ F_{p1} & F_{p2} & \cdots & F_{pp} \end{bmatrix}$$

where F_{ii} is square and irreducible. (note- if F_{ii} is a scalar, it is equal to 0.)

F is said to be lower triangularly complete if in every row *i* there exists at least one $j < i$ such that $F_{ij} \neq 0$ (i.e. it has least one nonzero entry).

F is said to be lower triangularly positive if $F_{ij} > 0, \forall j < i$

F is lower triang. Complete iff the associated graph has a spanning tree

Diagonal Dominance

Matrix $E = [e_{ij}] \in R^{n \times n}$ is diagonally dominant if, for all *i*, $e_{ii} \geq \sum_{j \neq i} |e_{ij}|$

It is strictly diagonally dominant if these inequalities are all strict.
E is strongly diagonally dominant if at least one of the inequalities is strict [Serre 2000]
E is irreducibly diagonally dominant if it is irreducible and at least one of the inequalities is strict.

Let *E* be a diagonally dominant M matrix (i.e. nonpositive elements off the diagonal, nonnegative elements on the diagonal).

Then $\lambda = 0$ is an eigenvalue of E iff all row sums are equal to 0.
Moreover, let *E* be irreducible with all row sums equal to zero. Then $\lambda = 0$ has multiplicity of 1.

Thm. Let *E* be strictly diagonally dominant or irreducibly diagonally dominant.
Then *E* is nonsingular.
If in addition, the diagonal elements of E are all positive real numbers, then

$$\mathrm{Re}\,\lambda_i(E) > 0,\ 1 \leq i \leq n$$

SO. Let graph be irreducible.
Then the Laplacian L has simple e-value at 0.
Add a positive number to any diagonal entry of L to get $\bar{L}$.
Then $\bar{L}$ is nonsingular and $-\bar{L}$ is stable.

Zhihua Qu Book 2009

Thm. Properties of Irreducible M matrices.

Let *E=sI-A* be an irreducible M matrix, that is, $A \geq 0$ and is irreducible.

Interpret E as the Laplacian matrix L

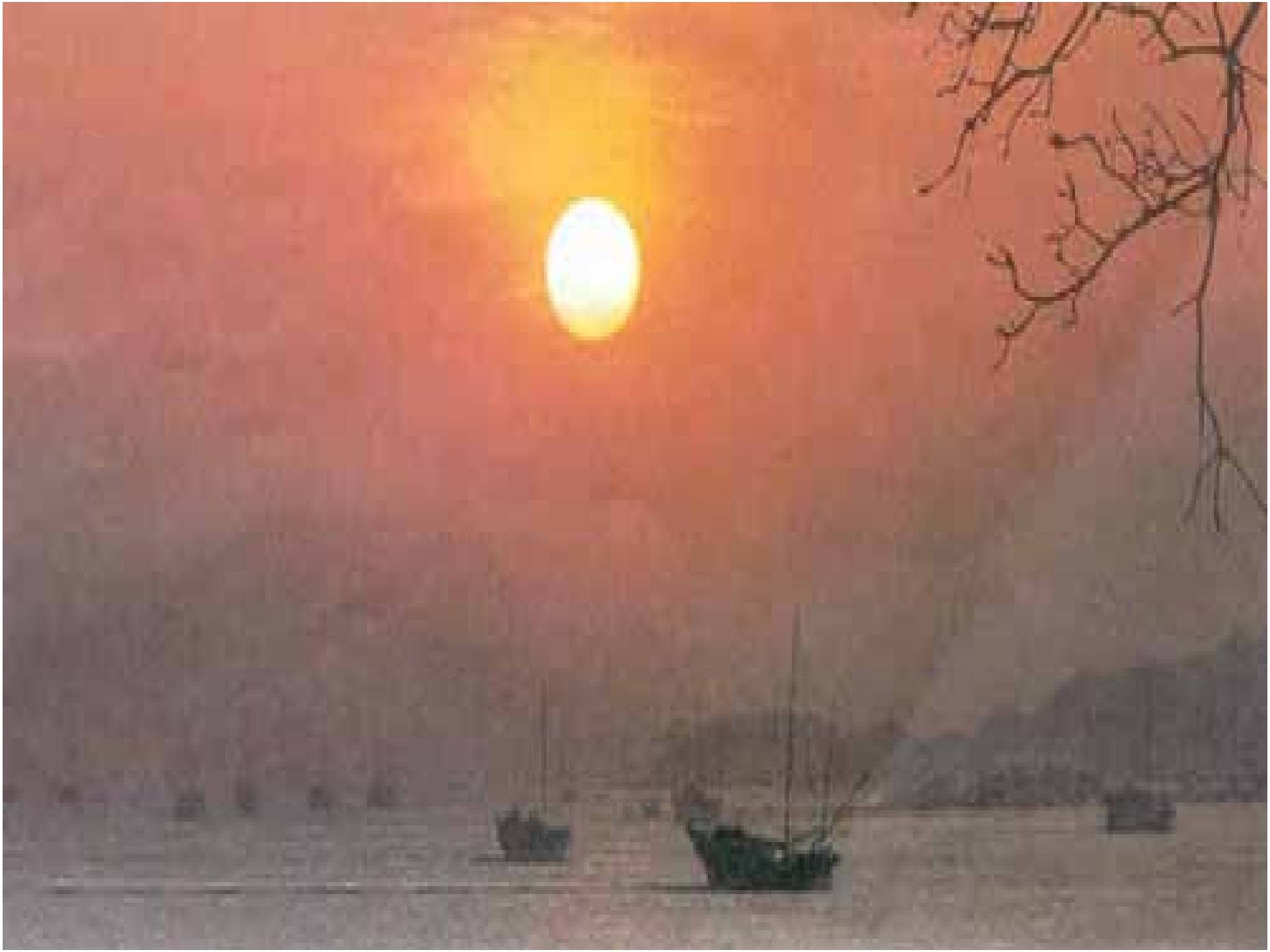
Then,

1. *E* has rank *n-1*.
2. there exists a vector *v>0* such that *Ev=0*.
3. $Ex \geq 0$ implies *Ex=0*.
4. Each principal submatrix of order less that *n* is a nonsingular M matrix
5. $(D+E)^{-1}$ exists and is nonnegative for all nonnegative diagonal matrices *D* with at least one positive element.
6. Matrix E has Property c.
7. There exists a positive diagonal matrix *P* such that

   $PE + E^T P$ is positive semidefinite,

   That is, matrix *E* is pseudo-diagonally dominant,
   That is, matrix *-E* is Lyapunov stable.

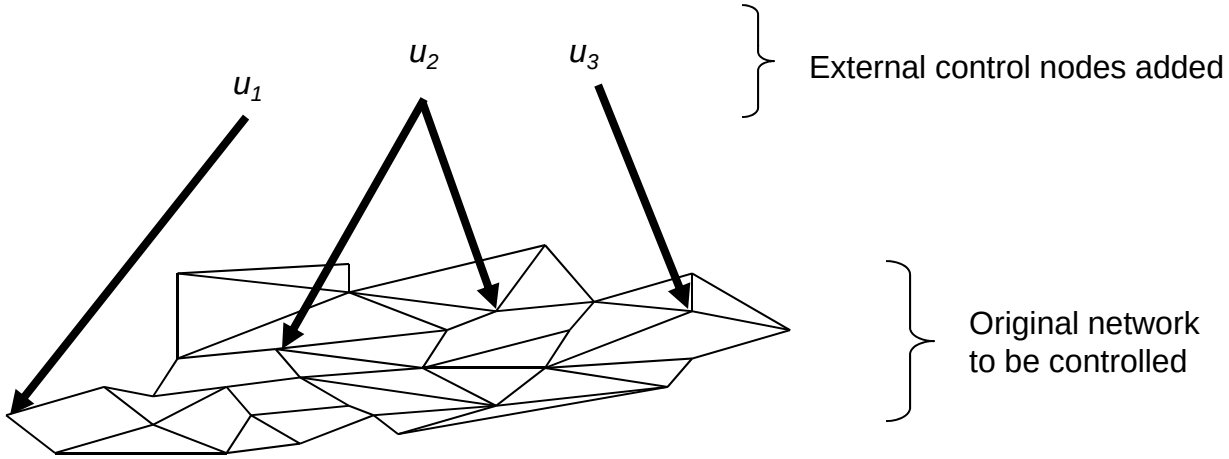
Used for Lyapunov Proofs



Lei Guo – Soft control

Do not change the local protocols of the nodes

can only add additional neighbors to influence existing nodes



How to pick
Injection nodes?

Extension of virtual leader approach

Add m additional control nodes

$B = [b_{ij}] \geq 0$ where $b_{ij} \geq 0$ is the weight from control node u_j to existing network node v_i.

Augmented state

$$\{\bar{x}_i : i = 1, N+m\} = \{x_1, x_2, \ldots, x_N, u_1, \cdots, u_m\}$$

New connectivity matrix

$$\bar{A} = \begin{bmatrix} A & B \end{bmatrix} = \left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1N} & b_{11} & b_{12} & \cdots & b_{1m} \\ a_{21} & \ddots & & \vdots & b_{21} & \cdots & & b_{2m} \\ \vdots & & & & \vdots & & & \\ a_{N1} & a_{N2} & \cdots & a_{NN} & b_{N1} & \cdots & & b_{Nm} \end{array}\right] \equiv [\bar{a}_{ij}] \in R^{N \times (N+m)}$$

Row sum = $\bar{d}_i$

Laplacian

$$\underbrace{\begin{bmatrix} \bar{D} - A \end{bmatrix}}_{} \quad -B \Big] = \left[\begin{array}{cccc|cccc} \bar{d}_1 - a_{11} & -a_{12} & \cdots & -a_{1N} & -b_{11} & -b_{12} & \cdots & -b_{1m} \\ -a_{21} & \ddots & & \vdots & -b_{21} & \cdots & & -b_{2m} \\ \vdots & & & & \vdots & & & \\ -a_{N1} & -a_{N2} & \cdots & \bar{d}_N - a_{NN} & -b_{N1} & \cdots & & -b_{Nm} \end{array}\right]$$

$$\bar{L} = \bar{D} - A$$

Modified local voting protocol

$\dot{x}_i = \mu_i$ Lei Guo

$\mu_i = \sum_{j \in N_i} \bar{a}_{ij}(\bar{x}_j - \bar{x}_i)$ SOFT CONTROL, includes new control nodes in some nbhds

modified diagonal matrix of in-degrees

$\bar{D} = diag\{\bar{d}_i\}$

with $\bar{d}_i = \sum_{j=1,N+m} \bar{a}_{ij}$ the *i*-th row sum of $\bar{A}$, which includes the new control nodes.

$\bar{d}_i = d_i + \delta_i$ where δ_i = *i*-th row sum of control matrix *B*.

Modified Laplacian

$\bar{L} = \bar{D} - A \in R^{N \times N}$

New closed-loop system

$\dot{x} = -\bar{L}x + Bu$

Row sum of $[\bar{L} \quad -B] = \left[\bar{D} - A \quad -B\right]$ is zero

But i-th row sum of $\bar{L} = \bar{D} - A$ has been increased by δ_i

$$\bar{L} = \bar{D} - A = \Delta + L \equiv \begin{bmatrix} \delta_1 + L_{11} & L_{12} & \cdots & L_{1N} \\ L_{21} & \delta_2 + L_{22} & & \vdots \\ \vdots & & \ddots & \\ L_{N1} & L_{N2} & \cdots & \delta_N + L_{NN} \end{bmatrix} \equiv [\bar{L}_{ij}] \in R^{N \times N}$$

δ_i = i-th row sum of control matrix B.

Lemma Let L have row sum zero and be irreducible.
At least one $\delta_i \neq 0$

Then

$$\bar{L} = \bar{D} - A = \Delta + L \equiv \begin{bmatrix} \delta_1 + L_{11} & L_{12} & \cdots & L_{1N} \\ L_{21} & \delta_2 + L_{22} & & \vdots \\ \vdots & & \ddots & \\ L_{N1} & L_{N2} & \cdots & \delta_N + L_{NN} \end{bmatrix} \equiv [\bar{L}_{ij}] \in R^{N \times N}$$

is irreducibly diagonally dominant and hence nonsingular.

Lemma. Then $-\bar{L}$ is asymp. stable.

$$\bar{L} = \bar{D} - A = \Delta + L \equiv \begin{bmatrix} \delta_1 + L_{11} & L_{12} & \cdots & L_{1N} \\ L_{21} & \delta_2 + L_{22} & & \vdots \\ \vdots & & \ddots & \\ L_{N1} & L_{N2} & \cdots & \delta_N + L_{NN} \end{bmatrix} \equiv [\bar{L}_{ij}] \in R^{N \times N}$$

δ_i = i-th row sum of control matrix B.

RTP 1. Let $\bar{L} = diag\{\delta_i\} + L, \quad \delta_i \geq 0$ and L be irreducible

Then relate eigenvalues of $\bar{L}$ to those of L . They are shifted right by some amount.

Special case. $\delta_i = c, \forall i$

Then all e-vals are shifted right by c

Define $\Delta = diag\{\delta_i\}$

Conjecture. Let L be irreducible. Then $\|\Delta + L\| > \|\Delta\|$

My Theorem

Define $\Delta = diag\{\delta_i\}$

Let

$$\bar{L} = \bar{D} - A = \Delta + L \equiv \begin{bmatrix} \delta_1 + L_{11} & L_{12} & \cdots & L_{1N} \\ L_{21} & \delta_2 + L_{22} & & \vdots \\ \vdots & & \ddots & \\ L_{N1} & L_{N2} & \cdots & \delta_N + L_{NN} \end{bmatrix} \equiv [\bar{L}_{ij}] \in R^{N\times N}$$

Then the determinant of $\bar{L} \in R^{N\times N}$ is given by

$$\left|\bar{L}\right| = \left|\Delta + L\right| = \left|L\right| + \sum_{s=1}^{k} \sum_{1\le j_1 < j_2 < \cdots < j_s \le k} \left(\delta_{i_{j_1}} \delta_{i_{j_2}} \cdots \delta_{i_{j_s}}\right) \bar{L}\left(\frac{i_{j_1}\, i_{j_2} \cdots i_{j_s}}{i_{j_1}\, i_{j_2} \cdots i_{j_s}}\right)$$

Minor with rows and columns struck out

Example- one diagonal entry δ_i positive

$$\left|\bar{L}\right| = \left|\Delta + L\right| \equiv \begin{vmatrix} L_{11} & L_{1i} & \cdots & L_{1N} \\ L_{i1} & \delta_i + L_{ii} & & \vdots \\ \vdots & & \ddots & \\ L_{N1} & L_{Ni} & \cdots & L_{NN} \end{vmatrix} = \begin{vmatrix} L_{11} & L_{1i} & \cdots & L_{1N} \\ L_{i1} & L_{ii} & & \vdots \\ \vdots & & \ddots & \\ L_{N1} & L_{Ni} & \cdots & L_{NN} \end{vmatrix} + \begin{vmatrix} L_{11} & 0 & \cdots & L_{1N} \\ L_{i1} & \delta_i & & \vdots \\ \vdots & & \ddots & \\ L_{N1} & 0 & \cdots & L_{NN} \end{vmatrix}$$

$$= \left|L\right| + \delta_i L\left(\frac{i}{i}\right)$$

To increase the determinant as much as possible-
Add δ_i to the node with the largest OUT-degree, i.e. largest column sum

We call the out-degree of node *i* its *influence* or *social standing*.

- John Baras

Overall Dynamics

$$\begin{bmatrix}\dot{x}\\ \dot{u}\end{bmatrix}=\begin{bmatrix}-\bar{L} & B\\ 0 & 0\end{bmatrix}\begin{bmatrix}x\\ u\end{bmatrix}$$

$$L^{aug}=\begin{bmatrix}\bar{L} & -B\\ 0 & 0\end{bmatrix}=\begin{bmatrix}\bar{D} & 0\\ 0 & 0\end{bmatrix}-\begin{bmatrix}A & B\\ 0 & 0\end{bmatrix}$$

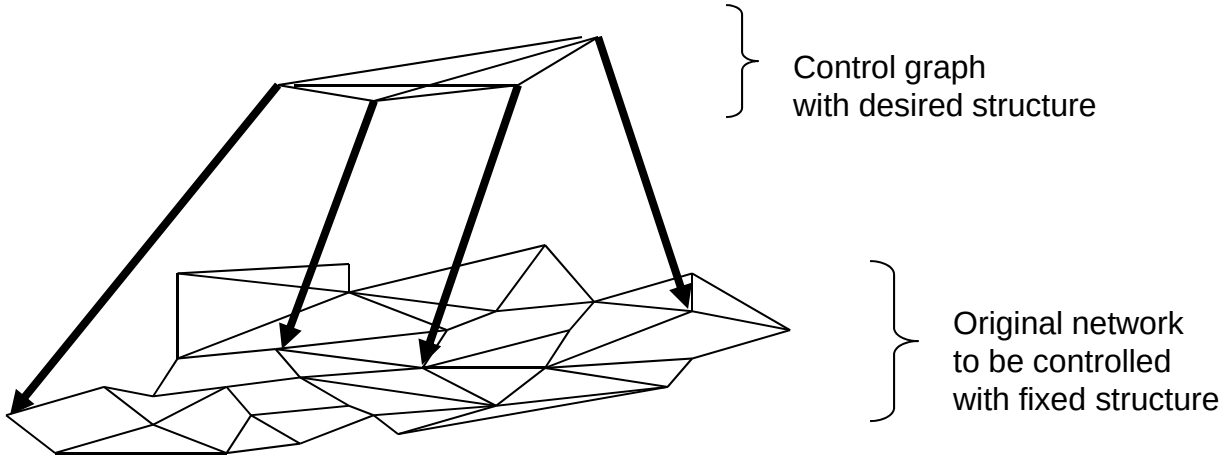
has m e-vals at 0

Does not reach consensus unless matrix is irreducible.

L^{aug} irreducible iff m=1

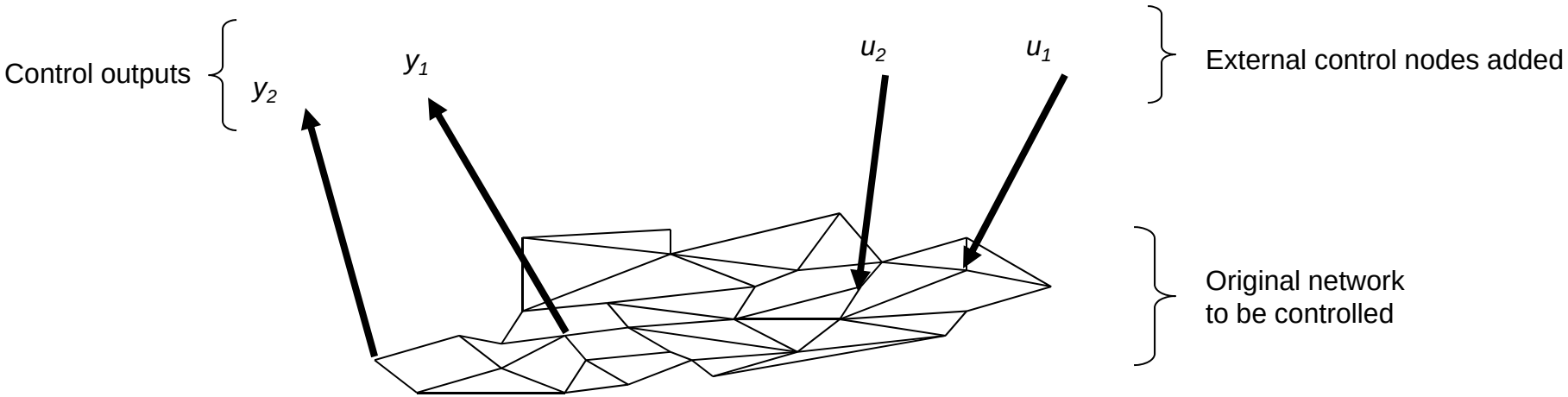
Add control graph $L^{aug}=\begin{bmatrix}\bar{L} & -B\\ 0 & L^G\end{bmatrix}=\begin{bmatrix}\bar{D} & 0\\ 0 & D^G\end{bmatrix}-\begin{bmatrix}A & B\\ 0 & G\end{bmatrix}$

$$\begin{bmatrix}\dot{x}\\ \dot{u}\end{bmatrix}=\begin{bmatrix}-\bar{L} & B\\ 0 & -L^G\end{bmatrix}\begin{bmatrix}x\\ u\end{bmatrix}$$



Induced Strogatz Small World Structure

Reduced diameter= longest path length, larger Fiedler e-val, so faster



Results. To make the controlled network as fast as possible,

Tap into the node with the LARGEST out-degree (highest social standing)

And take measured outputs from the nodes with the SMALLEST out-degree

- Zhihua Qu

