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Graph Theory

Graph theory is the branch of mathematics which deals with entities and their mutual relationships. The entities are represented by nodes or vertices and the existence of the relationship between nodes is represented as edges between/among the nodes. These nodes and relationships could be of a variety of nature like the nodes could be humans and edges could be the friendship between any two of them, nodes could be army officers and edges could be chain of command between them, nodes could be various destinations and edges could be roads between two of them, nodes could be cells in a cellular network and edges between two nodes could represent that they are adjacent, nodes could be atoms of Carbon, Oxygen and Hydrogen and edges representing covalent bonds between them, nodes could be critical points in a piping systems and edges could be pipes between them, nodes could be electrical components and edges could be connections between them, nodes could be states of a Markov process and edges could be possible transition between them and nodes could be various agents in a distributed system and edges could be the flow of information between them. These examples cover a wide range of walks of life including computer science, telecommunication, organic chemistry, distributed control systems and many others. Depending upon the application the nature of edges could be different. The edges could be simple non directional edges as incase of friendship between two persons and covalent bond between two atoms, they could be directional edges as in case of chain of command, there could be self edges as in case of Markov chains, there could be multiple edges between nodes as incase of two different roads between two destinations, there could be weighted edges as in case of road between two destinations weighted by the distance between two destinations or there could be the concept of hyper edges as incase of friendship groups. In these applications graph theory is used to address an endless list of problems. Just to name a few of them are shortest distance between two destinations, routing in computer networks, allocation of frequencies to cellular networks, determination of organic polymers, determination of minimum points to isolate two geographic regions, determination of flow between two nodes, analysis of electrical networks, design and analysis of data structure, distributed algorithms, parallel processing, load balancing and consensus in distributed networks.

Depending upon the application, the graph theory deals with the various types of graphs like undirected, directed, simple, weighted and hyper graphs. To begin with let's choose the simple finite graphs and come to the formal definition of a simple graph, these simple graphs would be called graphs in our discussion.

1.1 Graph:

A graph is an ordered pair G = (V,E) such that $E \subseteq [V]^2$ where $[V]^k$ is the set containing all the subsets of V with k elements, these subsets of V are call k-subsets of V. The cardinality of $[V]^k$ is represented as $|[V]^k|$ and equal to ${}^{|V|}C_k$ where |V| is the cardinality of V called the order of a graph and will be represented by n in our discussion. Connecting this definition with the applications as discussed earlier,

V is called the set of vertices/nodes and E is called the set of edges. Consequently the elements of V are called vertices of the graph and usually denoted by v and the elements of E are called edges of the graph and usually denoted by e. Also referring to a graph G its order can also be represented as |G| (=|V|) and its number of edges can also be represented as ‖G‖ (=|E|). It must be noted here that a simple graph may not contain self loops, multiple edges. Also all their edges are undirected, connecting two vertices and do not have any weight.

A vertex v is incident with an edge e if v ∈ e or we may say v is an end-vertex of the edge e, an edge connects its two end-vertices, and e is said to be an edge at v. If X and Y are two subsets of V then E(X,Y) represents the subset of E incident in both X and Y. Two vertices are neighbors/adjacent in G if there is an edge between them. Similarly two edges are adjacent if they have a common vertex. On contrary a set of vertices or edges is call independent of if they are pair wise non adjacent.

Two vertices with an edge between them are called neighbors of each other while two vertices reachable from one another through a sequence of vertices and edges are called connected. A graph is called connected if every node of the graph is connected to every other node. A graph which is not connected is called a disconnected graph. A connected graph is said to have one component while a disconnected graph is the union of more than one connected components.

The number of edges incident on a node v is called its degree and denoted by d(v). The mean degree of a node in a graph is denoted by d(G) and given by

$$d(G) = \frac{1}{|V|}\sum_{v \in V} d(v)$$

Also the minimum of all the degrees taken over all the vertices is called minimum degree of the graph and denoted by δ(G). Similarly maximum of all the degrees taken over all the vertices is called maximum degree of the graph and denoted by Δ(G). These definitions directly imply the following inequality

$$\delta(G) \leq d(G) \leq \Delta(G)$$

Now let's come to a pretty straightforward but a famous result in graph theory known as Handshaking Lemma.

Lemma 1.1.1 (The Handshaking Lemma)

The sum of degrees of all the nodes is equal to the number of edges in a graph. In its mathematical form we may write it as

$$\sum_{v \in V} d(v) = 2|E|$$

Proof:

If we sum all the degrees in a graph then every edge is counted twice and this completes our proof to this lemma. ■

Another entity related to graphs is number of edges of G per vertex and denoted by ε(G), mathematically we may write

$$\varepsilon(G) = |E| / |V| = \frac{1}{2|V|}\sum_{v \varepsilon V} d(v) = \frac{1}{2} d(G)$$

The above equations are written with the help of the handshaking Lemma and the previously defined entities.

There are some interesting results which are directly followed from the preliminaries.

Lemma 1.1.2

The number of vertices of odd degree is even.

Proof:

From the Handshaking Lemma sum of degrees of all the vertices are even. It is not possible unless number of vertices with odd degree is even. ■

Lemma 1.1.3

In every graph with more than 1 vertex there are at least two nodes with the same degree.

Proof:

If there are two or more vertices with zero degree (isolated vertices) we are done, otherwise there is a connected component in the graph with k (1< k≤ n) vertices. In this connected component there is a choice for each of k vertices to have any degree from 1 to k-1. And by Pigeon Hole Principle at least two vertices must have the same degree. ■

Lemma 1.1.4

If $\delta(G) > (n-2)/2$ $(n>1)$ then the graph is connected.

Proof:

Let the graph is not connected and it has k (k>1) components. Under the given condition each component must be having more than n/2 nodes →n>k (n/2). This is not possible for any value of k.

■

1.2 More on Graphs

There are some special types of graphs we refer most often. The most famous of these is a complete graph. A graph G is called a complete graph if $E(G) = [V]^2$, and represented as K^n. If all the vertices of a graph are of the same degree then such a graph is called k-regular. A graph G is called a bipartite graph if V(G) can be partitioned in two sets X and Y such as there is no edge whose both of the edges are in the same partition. We can similarly define an r-partite graph. The notion can be further extended to a complete r-partite graph. A complete r-partite graph contains all the possible edges for a given partition, if $V(G) = \bigcup_{i=1}^{r} V_i$ and $V_j \bigcap_{j \neq l} V_l = \phi$ then the corresponding r-partite graph can be represented as $K_{V_1, V_2, \ldots, V_r}$. A complete bipartite graph $K_{1, n-1}$ is called a star and

represented as S_{n-1}. A graph is called a chain/path if all of its vertices can be ordered in such a way that they are connected iff they are consecutive in the order. The graph obtained by adding an edge between two vertices of degree 1 in a chain is called a cycle and represented as C_n. This is probably a suitable point where we may formally define a path within a graph. A sequence of distinct vertices stating from a vertex u to another vertex v such as all the consecutive vertices are adjacent in G is called a path xPy from u to v within G. If there exists an edge e= {u,v} then xPv U e is called a cycle within G. A graph having a cycle as a subgraph is called a cyclic graph. While an acyclic graph is called a forest and an acyclic connected graph is called a tree. Like path there is another thing defined is graphs and it is a walk. A walk is like a path with permission of repetition of vertices. A graph is called a planner graph if it can be embedded in a plane in such a way that its edges intersect only at its vertices. A planer graph divides the plane in several parts known as faces. In these faces one is always unbounded and extending beyond the vertices and edges of the graph is called external face and also counted among the faces of the planer graph.

The notion of sub-graph is very similar to the notion of subsets except that in case of graphs it must be extended to both the set of vertices and the set of edges. There are some interesting refinements in this notion. The first one is known as induced subgraph. If $G' \subseteq G$ and E(G') contain all the edges $xy \in E(G)$with $x, y \in V(G')$ then G' is called an induces subgraph of G. We also say that V(G')=V' induces or spans G' in G and write it as $G' \triangleq G[V']$. The second one is spanning subgraph, if $G' \subseteq G$ and V(G')=V(G) then G' is known as a spanning subgraph of G. In particular if the spanning subgraph is a tree then it is called a spanning tree of the graph. The third one is the subgraph obtained by deleting an edge e of the graph G and written as G\e. The fourth one is obtained by contracting an edge e of the graph G and written as G/e. Please note that the direction of slash makes a difference in these notations also the detailed discussion on the last two will follow in the later part of this text. The fifth one is due to the deletion of a vertex v of G, written as G-v which is obtained from G by deleting the vertex v and edges incident on it.

Several relationships could be defined in graphs; some of them are mentioned here. The notions of union and intersection in a graph are very much similar to the respective notions in the set theory with the only detail that for graphs the respective operations must be extended to the set of vertices and edges. Next, for a graph G(V,E), its compliment is denoted by $\overline{G}$ $(V, \overline{E})$ where $\overline{E} = [V]^2\text{-}E$. Two graphs G and G' are said to be isomorphic if there exists a bijection $\varphi: V(G) \rightarrow V(G')$ such as $xy \in E(G) \leftrightarrow \varphi(x)\ \varphi(y) \in E(G')$ and we write it as $G \simeq G'$.

For a graph G, there is another graph L(G) associated with it which is called line graph of G and obtained from G by presenting all the edges of G as the vertices in L(G) with edges between them iff they were incident on a common vertex in G. Similarly if we have a planer graph G then another graph G* obtained by expressing each face of G by a node in G* with edges between them iff corresponding faces in G had common edge. It must be noted that here we have an edge in G* against each edge in G which may cause to generate a multi graph. G * is called dual of G for the particular embedding.

Our knowledge about graphs up to this extent allows us to work with a little involved results in the graph theory.

Proposition 1.2.1

Every graph G with ‖G‖>0 has a subgraph H with δ(H)> ε (H) ≥ ε (G)

Proof:

Before going into the proof of the proposition we need to examine k-regular graphs. For these graphs (k>0), (G)=k and ε (G)=k/2, this shows that the proposition holds for such graphs.

Now if the graph G does not have any vertex whose degree is lesser than or equal to ε (G) the proposition already holds. However if there is a vertex with degree lesser than or equal to ε (G) we may plan to delete such vertex to get a new graph G_1. For this graph $|G_1|=|G|-1$ and $\|G_1\|=\|G\|-e_d$, where e_d is the number of edges deleted which due to the selection of the vertex is lesser than or equal to ε (G). This will cause a total reduction of $2e_d$ in the sum of degrees of all the nodes in G_1. We may say

$$\varepsilon(G)=\frac{\|G\|}{|G|}$$

and

$$\varepsilon(G_1)=\frac{\|G_1\|}{|G_1|}=\frac{\|G\|-e_d}{|G|-1}\ where\ e_d \le \varepsilon(G)$$

And it can be seen readily that under the given condition $\varepsilon(G_1) \ge \varepsilon(G)$. The process is carried on recursively getting G_{i+1} from G_i with $\varepsilon(G_{i+1}) \ge \varepsilon(G_i)$ every subsequent subgraph having greater value of $\varepsilon(G_i)$. The process is continued till we get the subgraph H = G_i with δ(H)> ε (H). Needless to mention that the process deemed to terminate before the trivial graph because and regular graph including K2 has already been seen to obey the proposition. ■

The above result gives us a procedure to find well connected nodes in a graph which are of great interest in many applications. The next result we are going to discuss relates to the (G) with the length of path and cycles within the graph.

Proposition 1.2.2

Every graph G contains a path of length δ(G) and a cycle of length at least δ(G)+1 provided δ(G)≥ 2.

Proof:

Let us consider a maximal path P={v_1, v_2,…,v_k} within the graph. As the path is maximal so all the neighbors of v_k must be present in P, and this completes the first part of the proposition. For δ(G) ≥ 2 there must be at least one neighbor of v_k other than v_{k-1} in P. Suppose that v_i is the neighbor of v_k with the least index in P, and we have just found a cycle {v_i, v_{i+1},…,v_k, v_i} with length greater than or equal to δ(G)+1. ■

To have further enjoyment we need to know a few more things about graphs. The first one to come is the girth of a graph which the length of minimum cycle presents in the graph, it is denoted by

g(G). And the second one is the circumference of the graph which is the length of maximum cycle present within the graph. For an acyclic graph its girth is defined as ∞ while its circumference is defined as 0. These apparently strange definitions are for the consistency of certain graph theoretical results for acyclic graphs.

Next are distance between nodes, diameter, radius and center of a graph. The minimum number of hops required to reach v_i from v_j or vice versa is known as distance between them and denoted as $d_G(v_i, v_j)$. If the two nodes are not reachable from one another the distance between them is defined as ∞. The maximum of the distance between any pair of nodes is known as the diameter of the graph. While minimum taken over all the nodes of their maximum distances from other nodes is called the radius of the graph and such node is called a center of the graph. The diameter of the graph G is denoted by diam(G) and its radius by rad(G).

Now we come to mutual relationships of these various graph parameters as defined above. These relationships are established in the form of the following lemmas.

The following pages may have some typos.

Lemma 1.2.3

For every cyclic graph G, $g(G) \leq 2diam(G)+1$.

Proof:

Let $g(G) > 2diam(G)+1$ → There must exist a pair of vertices on the minimum cycle with distance between them greater than or equal to diam(G)+1. This is in contradiction with the definition of diameter. ■

Lemma 1.2.4

For every graph G, $rad(G) \leq diam(G) \leq 2rad(G)$.

Proof:

For the disconnected graph both the diam(G) and rad(G) are infinite and the inequalities are trivial. Also for the connected graph the first inequality is trivial and evident from the definitions. For the second inequality for the connected graphs let's suppose x and y be two such vertices in G that $d_G(x, y) = diam(G)$. Now let u be the center of the graph → $d_G(x, u), d_G(u,y) \leq rad(G)$ → $d_G(x,y) \leq d_G(x, u)+ d_G(u,y) \leq 2rad(G)$ and this proves the second inequality and we are done. ■

Lemma 1.2.5

For a graph G with $rad(G) \leq k$ and maximum degree Δ ($\Delta \geq 3$) then $|V| < \Delta (\Delta-1)^k/(\Delta-2)$.

Proof:

Let u be a center of the graph, it may have utmost Δ neighbors. All these Δ neighbors may have utmost (Δ-1) neighbors which have not been counted yet. And going this way we may go utmost till the radius. We sum it up in the following mathematical expression.

$$|V| \leq 1+ (\Delta)+ (\Delta) (\Delta-1)+ (\Delta) (\Delta-1)^2 +\ldots+(\Delta) (\Delta-1)^{rad(G)-1}$$

For $\Delta \geq 3$

→$|V| \leq 1+(\Delta)((\Delta-1)^{rad(G)}-1)/(\Delta-2)$

→$|V| \leq \Delta((\Delta-1)^{rad(G)}/(\Delta-2)+1-\Delta/(\Delta-2)$

→$|V| < \Delta((\Delta-1)^{rad(G)}/(\Delta-2)$ and for $k \geq rad(G)$

→$|V| < \Delta((\Delta-1)^{K}/(\Delta-2)$ ■

Lemma 1.2.6

For a graph G with minimum degree $\boldsymbol{\delta}$ (≥ 3) and girth g has at least $n_o(\boldsymbol{\delta}, g)$ vertices where

$$n_o(\delta, g) = \begin{cases} 1+\delta\dfrac{((\delta-1)^{\frac{g-1}{2}}-1)}{\delta-2} \text{ when } g \text{ is odd} \\ 2\dfrac{((\delta-1)^{\frac{g}{2}}-1)}{\delta-2} \text{ when } g \text{ is even} \end{cases}$$

Proof:

Let g be odd. Let us start from any vertex v. It needs to have at least $\boldsymbol{\delta}$ neighbors. Now each of these neighbors must have at least $\boldsymbol{\delta}$-1 distinct neighbors. And starting from v this process will go on for at least (g-1)/2 steps. We can thus write the following inequality

$|V| \geq 1+(\boldsymbol{\delta})+(\boldsymbol{\delta})(\boldsymbol{\delta}-1)+(\boldsymbol{\delta})(\boldsymbol{\delta}-1)^2+\ldots+(\boldsymbol{\delta})(\boldsymbol{\delta}-1)^{(g-3)/2}$

For $g \geq 3$, this is the least odd value for the girdth

→$|V| \geq 1+(\boldsymbol{\delta})((\boldsymbol{\delta}-1)^{(g-1)/2}-1)/(\boldsymbol{\delta}-2)$

→$|V| \geq n_o(\boldsymbol{\delta}, g)$

And we are done for the odd girth. Now let g be even. Now starting from two adjacent vertices u and v, each of them needs to have at least ($\boldsymbol{\delta}$-1) neighbors other than u and v, and similarly each of these neighbors must have at least $\boldsymbol{\delta}$-1 distinct neighbors. However, this time starting from v this process will go on for at least (g-2)/2 steps. We can thus write the following inequality

$|V| \geq 2+2(\boldsymbol{\delta}-1)+2(\boldsymbol{\delta}-1)^2+\ldots+2(\boldsymbol{\delta}-1)^{(g-2)/2}$

For $g \geq 4$, this is the least even value for the girth

→$|V| \geq 2((\boldsymbol{\delta}-1)^{g/2}-1)/(\boldsymbol{\delta}-2)$

→$|V| \geq n_o(\boldsymbol{\delta}, g)$

And we are done ■

Now we come to the following proposition which gives us a very important result for the connected graphs.

Proposition 1.2.7

The vertices of a connected graph can always be enumerated as $\{v_1, v_2..., v_N\}$ such as $G_i=G[v_1,v_2..., v_i]$ is connected for every $i = 1,2..., N$.

Proof:

Let's start with any vertex v_1 within the enumeration. As a single to graph is trivially connected the statement is true for $i=1$. Now suppose that $G_k=G[v_1,v_2..., v_k]$ is connected. Now take any vertex v which has not been enumerated yet. Since the graph is connected, starting from v we can always reach an enumerated vertex, let v_j is the first one we reach following any suitable path. We enumerate the second last vertex as v_{k+1} in the path $v \sim v_j$, and we are inductively done as $G_{k+1} =G[v_1,v_2..., v_k,v_{k+1}]$ is connected. ■

As defined earlier an acyclic connected graph is a tree. Here are a few facts about the trees to further elaborate its structure.

Proposition 1.2.8

The following statements are equivalent.

i. T is a tree
ii. Any two vertices in T are connected by a unique path
iii. T is minimally edge connected
iv. T is maximally acyclic

Proof:

Let T be a tree.

→ T is connected and acyclic

→Any two vertices of T are connected by a unique path; if there are more than one path T would no longer acyclic.

→ (i→ii)

Let any two vertices in T are connected by a unique path

→ Deletion of any edge would disconnect at least two vertices

→T is minimally edge connected

→ (ii→iii)

Let T be minimally edge connected

→ Deletion of any on edge will render the graph disconnected.

→ T is acyclic

Also any two vertices x, y are connected in T

→ Addition of edge xy would create a cycle

→ T is maximally acyclic

→ (iii→iv)

Lastly let T is maximally acyclic

→ Addition of any xy edge would create a cycle

→ Any two vertices x and y are connected in T

→ T being connected and acyclic is a tree

→ (iv→i)

This completes our proof. ■

Proposition 1.2.9

The vertices of a tree T can always be enumerated as $\{v_1, v_2..., v_N\}$ such as v_i ∀ i=2... N has a unique neighbor in $\{v_1,v_2..., v_{i-1}\}$.

Proof:

From the Proposition 1.2.7 its well clear that vertices of any connected graph can be enumerated as $\{v_1, v_2..., v_N\}$ such as $G_i=G[v_1,v_2..., v_i]$ is connected ∀ i=1,2... N.

→ $G_i=T[v_1,v_2..., v_{i-1}]$ and $G_i=T[v_1,v_2..., v_i]$ ∀ i=2... N are both connected

→ v_i is has at least one neighbor in $\{v_1,v_2..., v_{i-1}\}$

Now for the uniqueness suppose that v_i has two neighbors in $\{v_1,v_2..., v_{i-1}\}$, let they be v_k and v_{k+j}. By Proposition 1.2.7 v_k and v_{k+j} are connected.

→ v_k and v_{k+j} and v_i will make a loop, which is in contradiction to the fact that T is a tree

→ v_i has a unique neighbor in $\{v_1,v_2..., v_{i-1}\}$ ∀ i=1,2... N ■

Proposition 1.2.10

A connected graph with N vertices is a tree if and only if it has N-1 edges.

Proof:

Let's have a tree T with N vertices. By the Proposition 1.2.9 the vertices of T can be enlisted as $\{v_1, v_2..., v_N\}$ such as each vertex in the list has a unique neighbor among the preceding vertices. Thus counting all the edges of a vertex to its preceding vertices these would be N-1 as the first one would not be having a preceding vertex.

Conversely suppose that T is a connected graph with N-1 edges. We use induction to establish this part. For N=2 the result is trivially true. Now let the result is true for N=k i.e. T, a connected graph with k vertices and k-1 edges is a tree. Now we add a vertex to this graph so that it remains a tree.

For this purpose we my either connect to an existing vertex or insert within an existing edge. In both the case only one edge would be added and we are done. ■

The next proposition gives us an important result for bipartite graphs.

Proposition 1.2.11
A graph is bipartite if and only if it has no odd cycles.

Proof:
Let's have a bipartite graph. It can be readily seen that there could not be an odd cycle.

Conversely let the graph has no odd cycle. To prove that it is bipartite we have to show that all of its components are bipartite. Thus without loss of generality we suppose that the graph is connected. Now enumerate the nodes as we did in Proposition 1.2.7 but this time with an additional step; each time we add a node we assign it a color either red or blue. With the first node assigned with red, we keep on assigning them color in such a way that each newly added node has a color different from it's already enumerated neighbors. Our claim is that it is always possible. It is trivial to see that under given condition of the absence of odd cycle it is always possible for some small number of nodes. Now inductively suppose that it is possible for us to enumerate/color k nodes in this way i.e. we have an enumeration $\{v_1, v_2 \ldots v_k\}$ such as $G_k = [v_1, v_2 \ldots v_k]$ is connected and with every node having a color different from its preceding neighbors. Now every node with the same (different) color in the enumeration is even (odd) hops apart. Now if we add a node v_{k+1} in this enumeration then it will have all the neighbors in $G_{k+1} = [v_1, v_2 \ldots v_k, v_{k+1}]$ having same color; else it would have odd cycles. So we may assign it with a color different from its neighbors in G_{k+1}. And placing the nodes with same color on one side we have shown a bipartition of the graph. ■

Now we come to probably the most famous notion in graph theory; Euler Tour. A closed walk in graph is called Euler tour if it contains every edge exactly once. A graph where it is possible to have an Euler tour is called Eulerian.

Theorem 1.2.12 (Euler)
A connected graph is Eulerian if and only if all of its vertices are of even degree.

Proof:
The necessity of an Eulerian graph to have all its vertices with even degree is evident from the fact that for each incoming edge to a vertex we need to have an outgoing edge to complete the Eulerian tour.

Conversely suppose that all the vertices of a connected graph have even degree. Now starting from any one node v_1 we would always come back to the same node as for every edge we enter into a node there would always be an unvisited edge to go out of that vertex except possibly for the initial vertex v_1. Now if we have already visited all the edges we are done. If not, then for our connected graph there need to have an already visited vertex v_2 with an unvisited edge. Following this edge, owing to the same reason as established earlier, we are again bound to come back to v_2. Continuing in the same way we would surely end at some vertex v_k, when we have visited all the edges. Now the sequence $v_1 \sim v_2 \sim \ldots \sim v_{k-1} \sim v_k \sim v_k \sim v_{k-1} \sim \ldots v_2 \sim v_1$ is our desired Eulerian tour. ■

Now we come to an important concept in graph theory, known as degree sequence. If we arrange the degrees of all the vertices of a graph in non increasing order then this sequence is known as the degree sequence of the graph. A question naturally arises that whether all the non increasing finite sequence of nonnegative integers corresponds to a graph. The answer to this question is No. A non increasing finite sequence of nonnegative integers is known as graphic if there is a graph with the given sequence as its degree sequence. There is a necessary and sufficient condition for a sequence to be graphic.

Theorem 1.2.13

A finite sequence of nonnegative integers $d= (d_1,d_2,...d_N)$ with $d_1 \geq d_2 \geq ... \geq d_N$ is graphic if and only if

$$\sum_{i=1}^{N} d_i \text{ is even and } \sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{N} \min(k, d_i) \; \forall k \leq N-1$$

Proof:

Let the given sequence of nonnegative integers $d= (d_1,d_2,...d_N)$ with $d_1 \geq d_2 \geq ... \geq d_N$ is graphic

→ There is a graph G with vertices having degrees $d_1,d_2,...d_N$

→ By using Lemma 1.1.1 (Hand Shaking Lemma) sum of the degrees of all the vertices will be even. This proves the first part of the necessary condition.

Now for the second part of the necessary condition

$$\sum_{i=1}^{k} d_i = \text{Sum of all the degree of } G_k[d_1, d_2, ...d_k] + \text{Number of edges from } \{v_1,v_2,...v_k\} \text{ to } V\text{-}\{v_1,v_2,...v_k\}$$

$$\sum_{i=1}^{k} d_i \leq 2\,k(k-1)/2 + \text{Number of edges from } \{v_1,v_2,...v_k\} \text{ to } V\text{-}\{v_1,v_2,...v_k\}$$

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{N} \min(k, d_i)$$

And this establishes the necessary condition.

Now for the sufficiency suppose that $\sum_{i=1}^{N} d_i \text{ is even and } \sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{N} \min(k, d_i) \; \forall k \leq N-1$

We will devise a method to make a graph corresponding to the given degree sequence under the given conditions.

...Work in progress

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