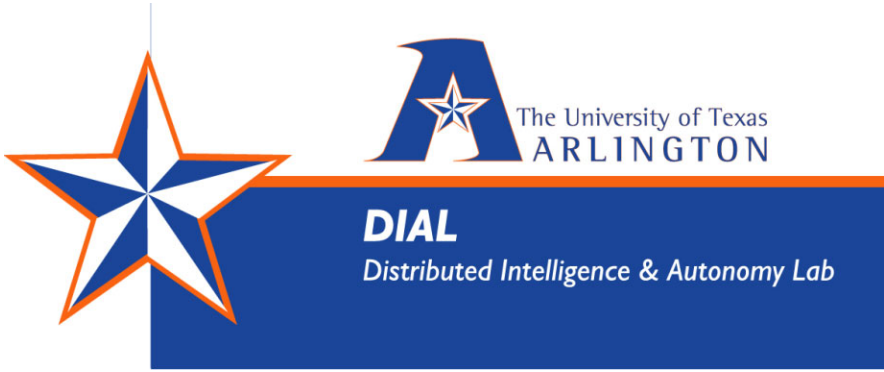
F.L. Lewis, NAI
Moncrief-O'Donnell Endowed Chair
Head, Controls & Sensors Group



UTA Research Institute (UTARI)
The University of Texas at Arlington

Multi-agent Consensus and Algebraic Graph Theory

Supported by NSF, ARO, AFOSR

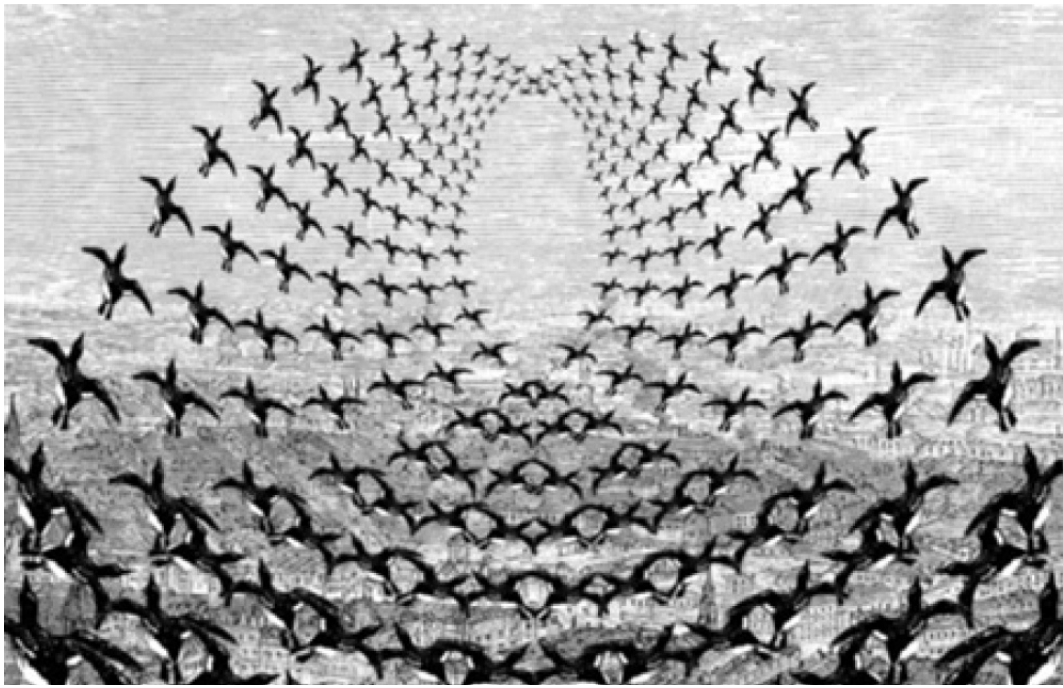


Motions of biological groups



Fish school

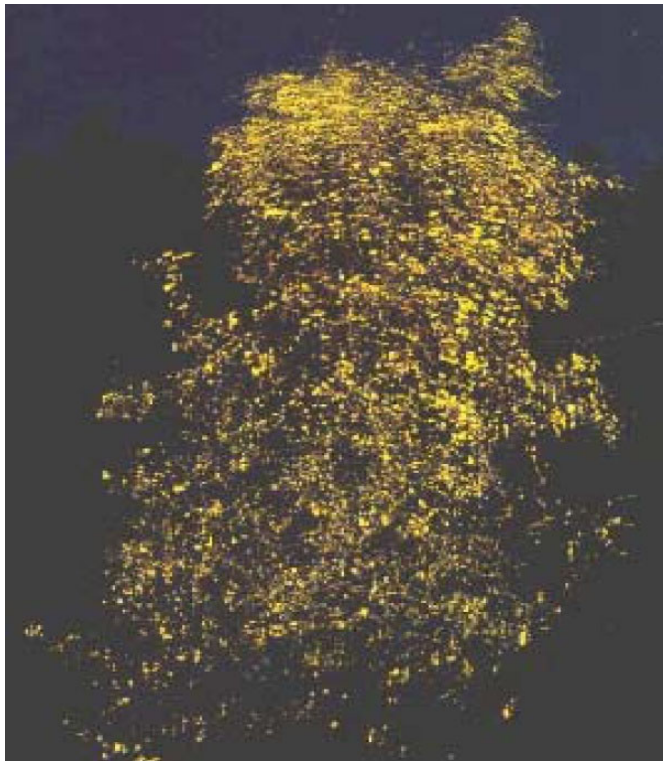
Birds flock



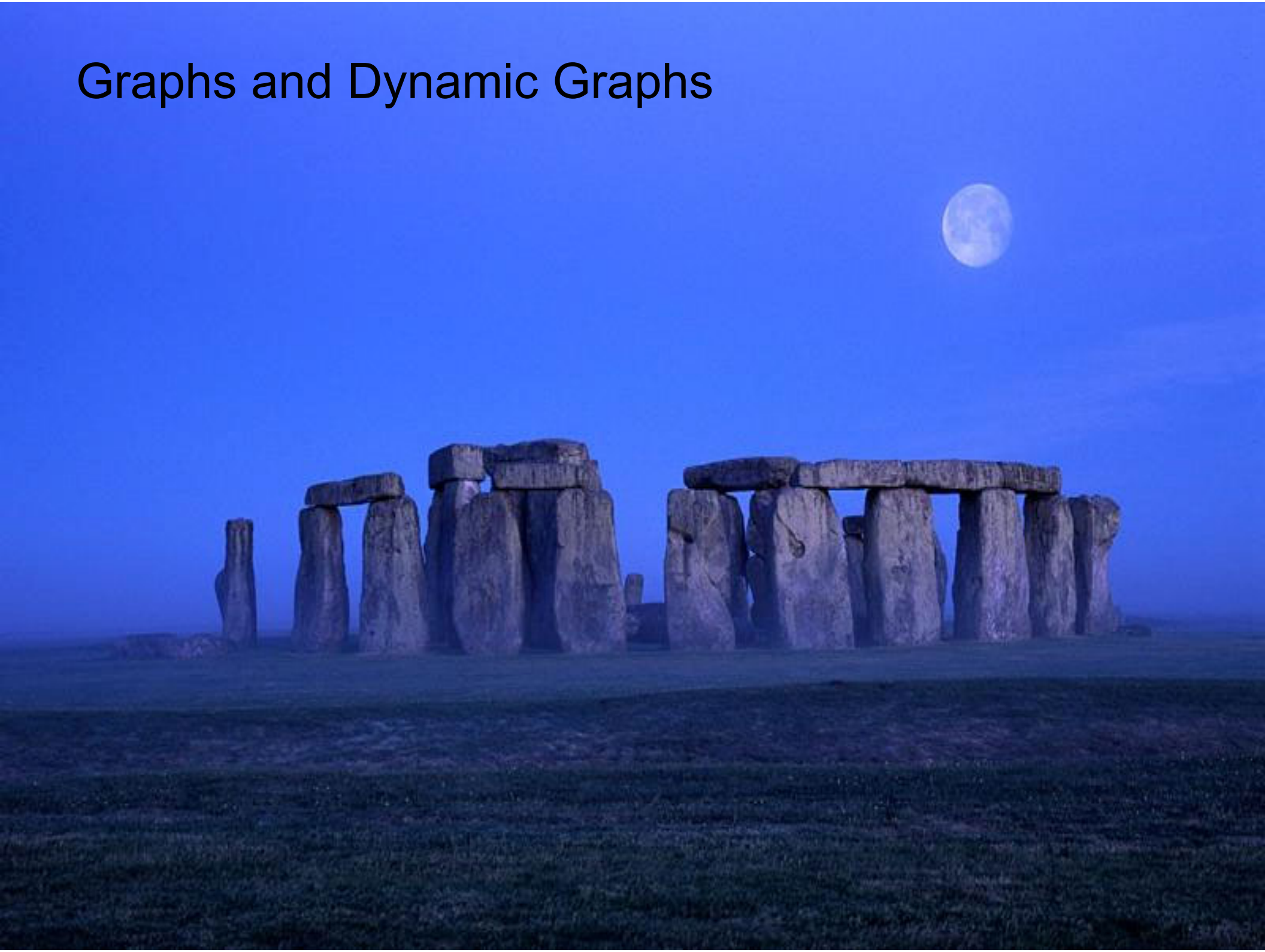


Locusts swarm

Fireflies synchronize



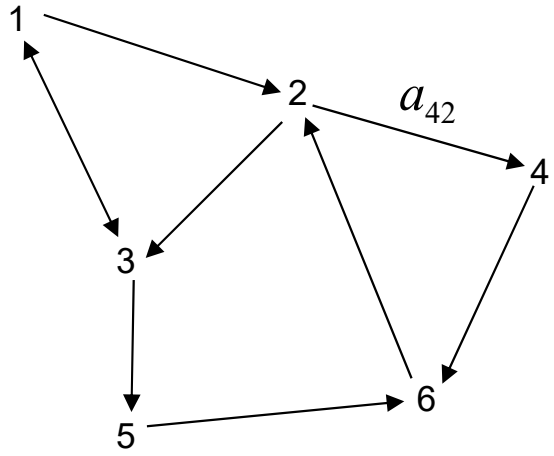
Graphs and Dynamic Graphs



Communication Graph – Algebraic Graph Theory

G= (V,E)

N nodes



Adjacency matrix

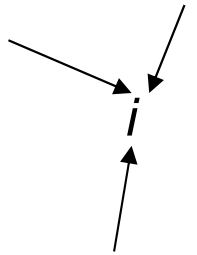
$$A = [a_{ij}]$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

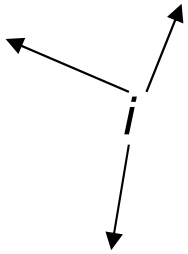
$$a_{ij} > 0 \; if \; (v_j, v_i) \in E$$

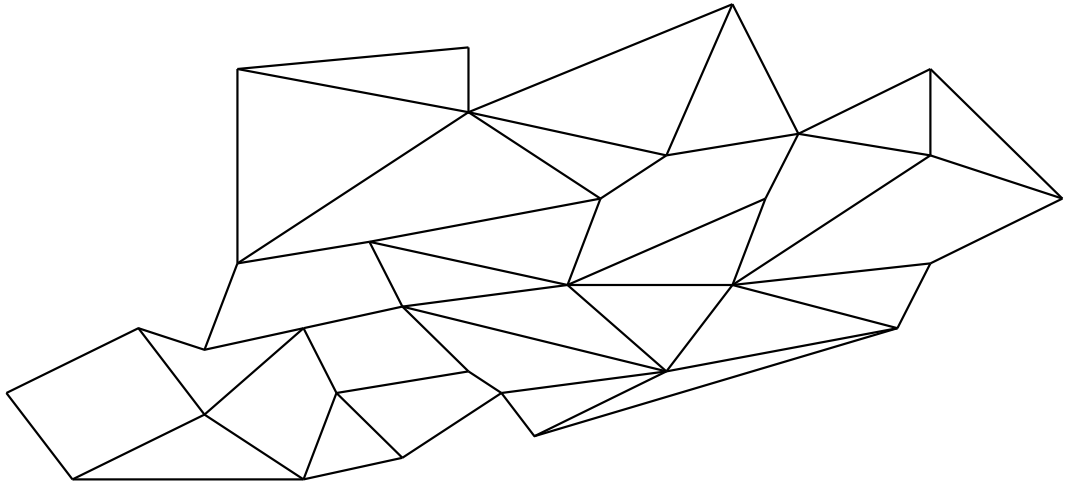
$$if \;\; j \in N_i$$

$d_i = \sum_{j=1}^{N} a_{ij}$ Row sum= in-degree N_i In-neighbors of node i



$d_i^o = \sum_{j=1}^{N} a_{ji}$ Col sum= out-degree N_o Out-neighbors of node i



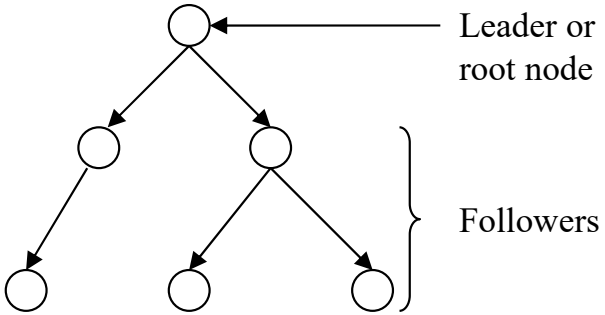


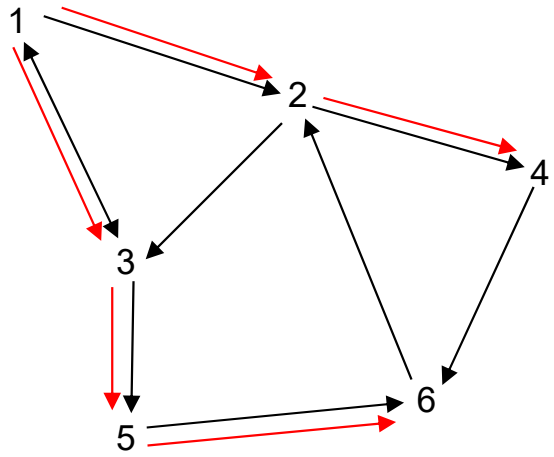
Strongly connected if for all nodes *i* and *j* there is a path from *i* to *j*.

Diameter= length of longest path between two nodes

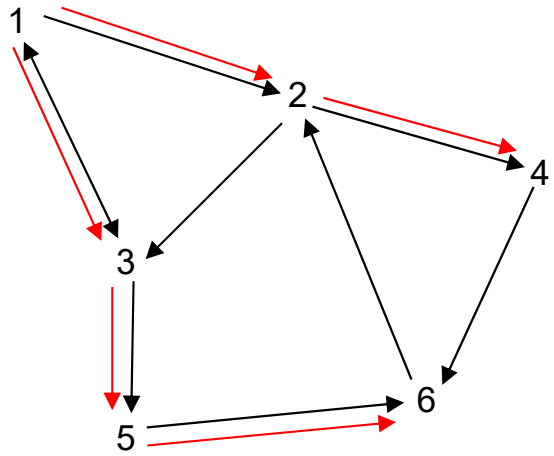
Volume = sum of in-degrees $Vol = \sum_{i=1}^{N} d_i$

Tree- every node has in-degree=1





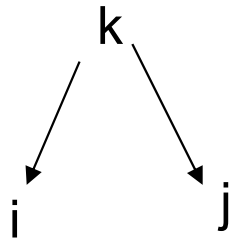
Spanning tree
Root node



Exists a spanning tree iff there is a node having a path to all other nodes

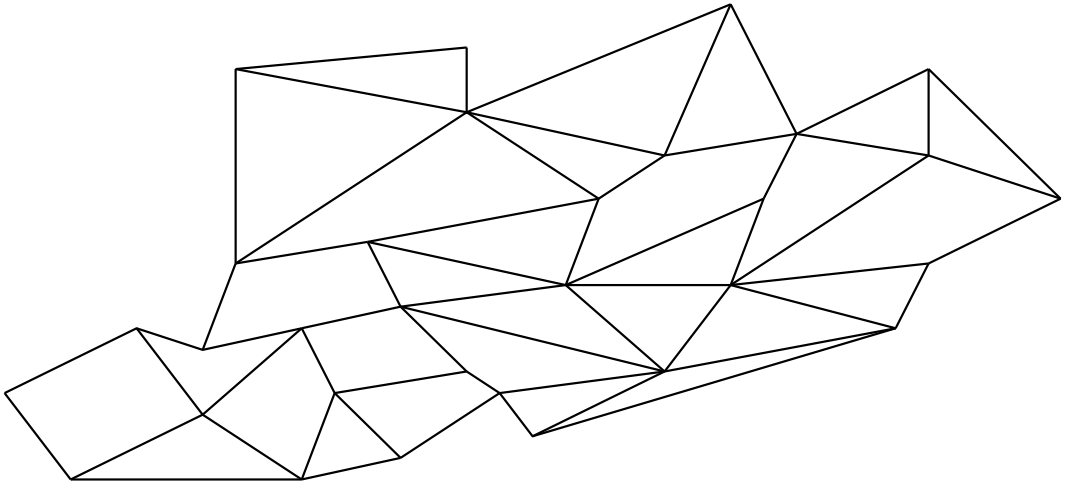
Graph strongly connected implies exists a spanning tree
Strongly connected implies Every Node is a root node

Quasi-strongly connected if for all nodes *i* and *j*
there exists a node *k* with a path to *i* and a path to *j*



Exists a spanning tree iff quasi-strongly connected

The Multi-agent Consensus Problem – Cooperative Control



$$\dot{x}_i = u_i$$

Agent dynamics $\dot{x}_i = u_i$

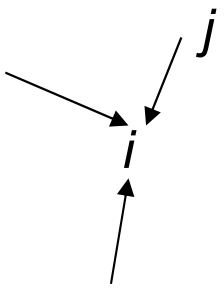
Find control u_i so that all agents reach the same steady-state value

$$x_i - x_j \to 0, \quad \forall i, j$$

Dynamic Graph- the Graphical Structure of Control
First-order Consensus of Multi-agent systems

Each node has an associated state $\dot{x}_i = u_i$

Standard Distributed local voting protocol $u_i = \sum_{j \in N_i} a_{ij}(x_j - x_i)$



Dynamics $\dot{x}_i = \sum_{j \in N_i} a_{ij}(x_j - x_i)$

$$u_i = -x_i \sum_{j \in N_i} a_{ij} + \sum_{j \in N_i} a_{ij} x_j = -d_i x_i + \begin{bmatrix} a_{i1} & \cdots & a_{iN} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$$

Global Vectors x

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} \qquad D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_N \end{bmatrix} \qquad A = [a_{ij}]$$

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} = -\begin{bmatrix} d_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & d_N \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & \cdots & a_{1N} \\ a_{21} & 0 & & \vdots \\ \vdots & & \ddots & \\ a_{N1} & \cdots & & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = -Dx + Ax$$

Dynamic Graph- the Graphical Structure of Control

First-order Consensus of Multi-agent systems

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_N \end{bmatrix} = u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} = -\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_N \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & \cdots & a_{1N} \\ a_{21} & 0 & & \vdots \\ \vdots & & \ddots & \\ a_{N1} & \cdots & & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = -Dx + Ax$$

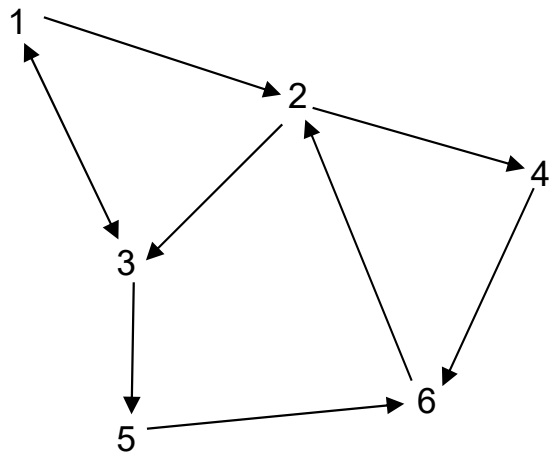
$$\dot{x} = u = -Dx + Ax = -(D - A)x = -Lx$$ *L*=*D*-*A* = graph Laplacian matrix

$$\dot{x}_i = \sum_{j \in N_i} a_{ij}(x_j - x_i)$$ Distributed Form of Closed-loop Consensus dynamics

$$\dot{x} = -Lx$$ Global Form of Closed-loop Consensus dynamics

If *x* is an *n*-vector then $\dot{x} = -(L \otimes I_n)x$

Communication Graph – Algebraic Graph Theory



Adjacency matrix

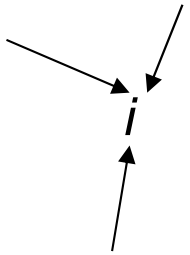
$$A = [a_{ij}]$$

$$a_{ij} > 0 \; if \; (v_j, v_i) \in E$$

$$if \;\; j \in N_i$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$d_i = \sum_{j=1}^{N} a_{ij}$ Row sum= in-degree N_i In-neighbors of node i



and the diagonal in-degree matrix and Laplacian are

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & -1 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$$

Note that the row sums of L are all zero.

Global Closed-loop system with local voting protocol

$$\dot{x} = -Lx \qquad x(t) = e^{-Lt}x(0)$$

L= graph Laplacian matrix

Global Closed-loop dynamics depends on L

$$L = D - A = \begin{bmatrix} d_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & d_N \end{bmatrix} - \begin{bmatrix} 0 & a_{12} & \cdots & a_{1N} \\ a_{21} & 0 & & \vdots \\ \vdots & & \ddots & \\ a_{N1} & \cdots & & 0 \end{bmatrix} \quad , \quad d_i = \sum_{j=1}^{N} a_{ij} \text{ = row sum}$$

L has row sum of 0 $\lambda_1 = 0$

$$0 = L\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = L\underline{1} = Lc\underline{1}$$ For any constant *c*

So at steady-state $0 = \dot{x} = -Lx = -Lc\underline{1}$ $x_{ss} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = c\underline{1}$

And all nodes reach the same value c

Multi-agent Cooperative Regulator

Dynamics $\dot{x}_i = u_i$

Cooperative Control $u_i = \sum_{j \in N_i} a_{ij}(x_j - x_i)$

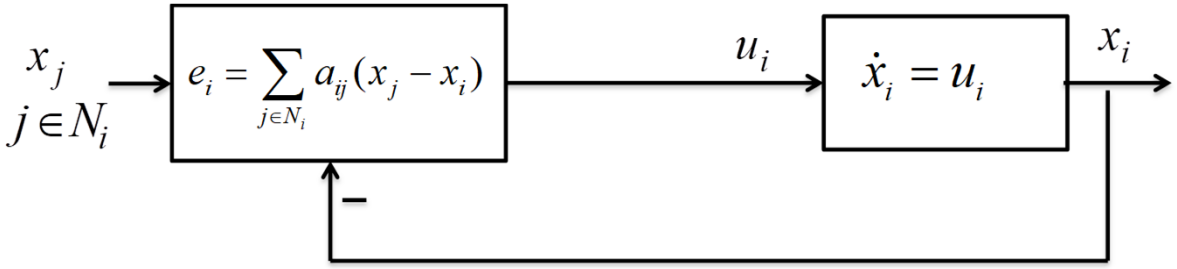
Distributed Closed-loop system

$$\dot{x}_i = \sum_{j \in N_i} a_{ij}(x_j - x_i)$$

Global Form

The same $\dot{x} = -Lx$

Local Neighborhood Consensus Error $e_i = \sum_{j \in N_i} a_{ij}(x_j - x_i)$



Multi-agent Cooperative Regulator

In the single-agent regulator problem, the state goes to zero.

In the multi-agent cooperative regulator problem,
the states go to a nonzero consensus value

Reynolds, Computer Graphics 1987

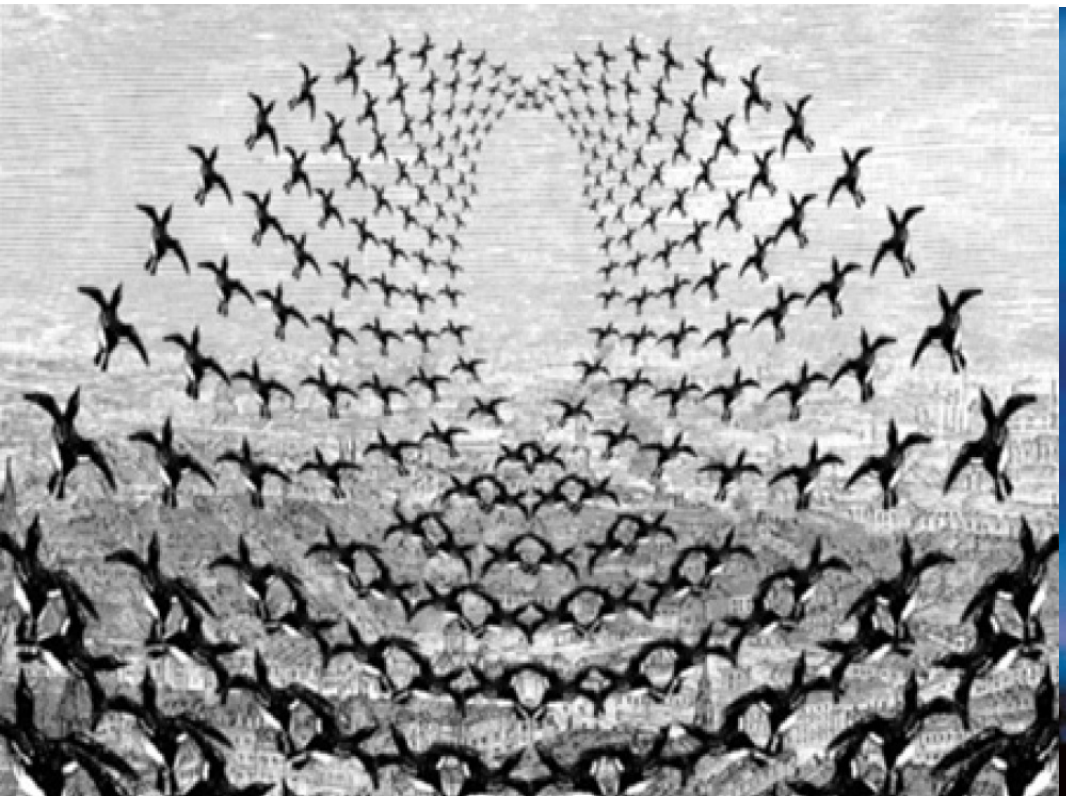
Flocking

Reynolds' Rules:

Alignment : align headings $\dot{\theta}_i = \sum_{j \in N_i} a_{ij}(\theta_j - \theta_i)$

Cohesion : steer towards average position of neighbors- towards c.g.

Separation : steer to maintain separation from neighbors





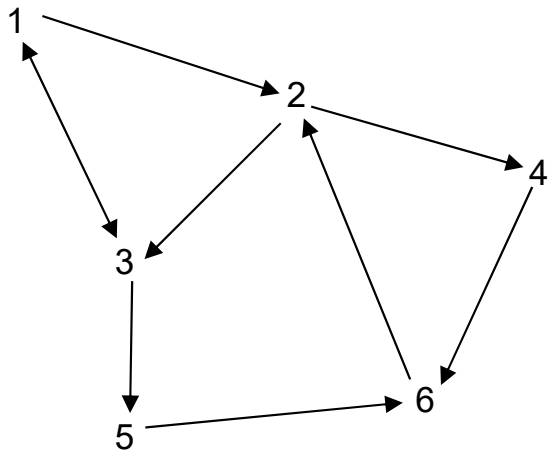
Distributed Adaptive Control for Multi-Agent Systems







Consensus Control for Swarm Motions

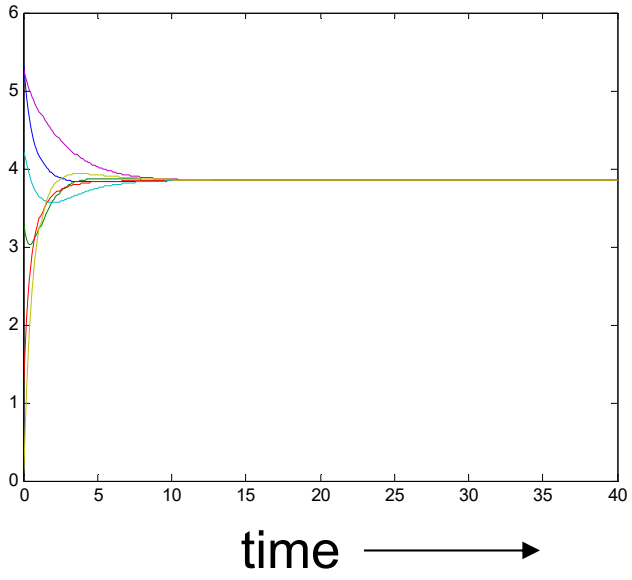


$$\dot{\theta}_i = \sum_{j \in N_i^c} a_{ij}(\theta_j - \theta_i)$$

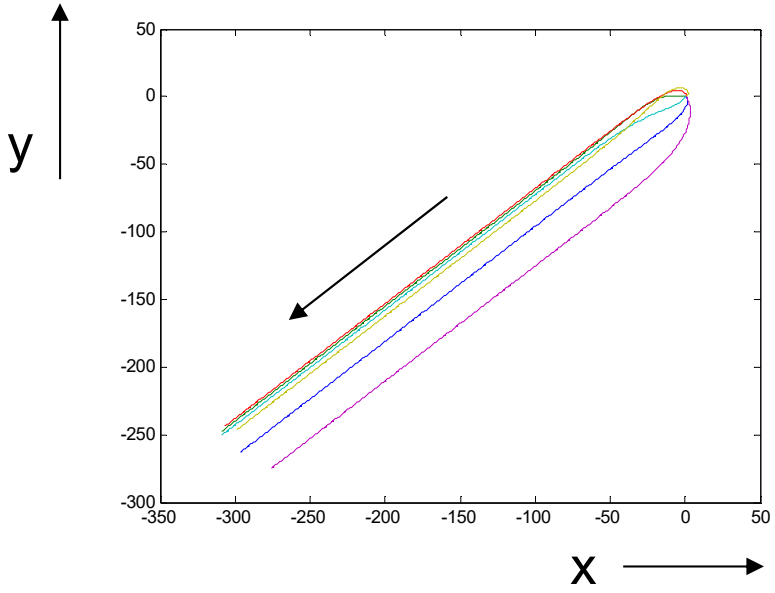
heading angle

$$\dot{x}_i = V \cos\theta_i$$

$$\dot{y}_i = V \sin\theta_i$$

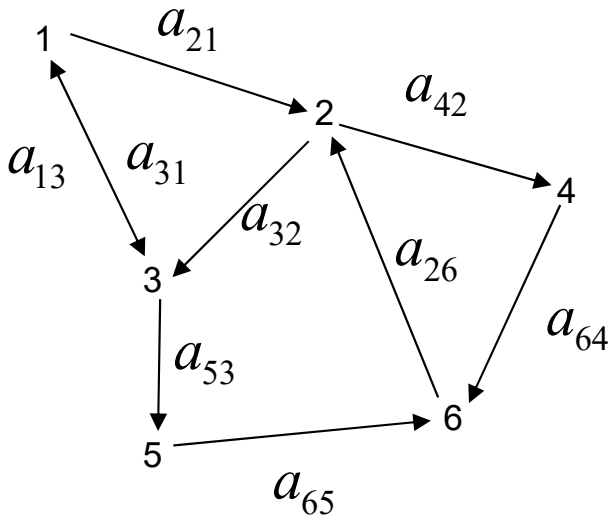


Convergence of headings



Nodes converge to consensus heading

Consensus Control for Swarm Motions



$$\dot{\theta}_i = \sum_{j \in N_i^c} a_{ij}(\theta_j - \theta_i)$$

heading angle

Alignment of Headings

Edge weight a_{ij}

$i \longleftarrow j$

Distributed Form of Consensus Law

$$\dot{\theta}_1 = a_{13}(\theta_3 - \theta_1)$$

$$\dot{\theta}_2 = a_{21}(\theta_1 - \theta_2) + a_{26}(\theta_6 - \theta_2)$$

$$\dot{\theta}_3 = a_{31}(\theta_1 - \theta_3) + a_{32}(\theta_2 - \theta_3)$$

$$\dot{\theta}_4 = a_{42}(\theta_2 - \theta_4)$$

$$\dot{\theta}_5 = a_{53}(\theta_3 - \theta_5)$$

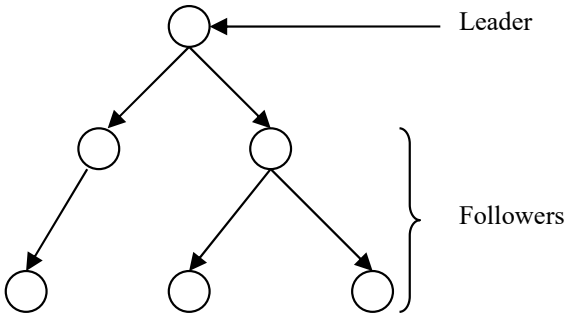
$$\dot{\theta}_6 = a_{65}(\theta_5 - \theta_6) + a_{64}(\theta_4 - \theta_6)$$

Global Form of Consensus Law

$$\dot{\theta} = -L\theta$$

θ Is a 6-vector

Consensus Control for Formations

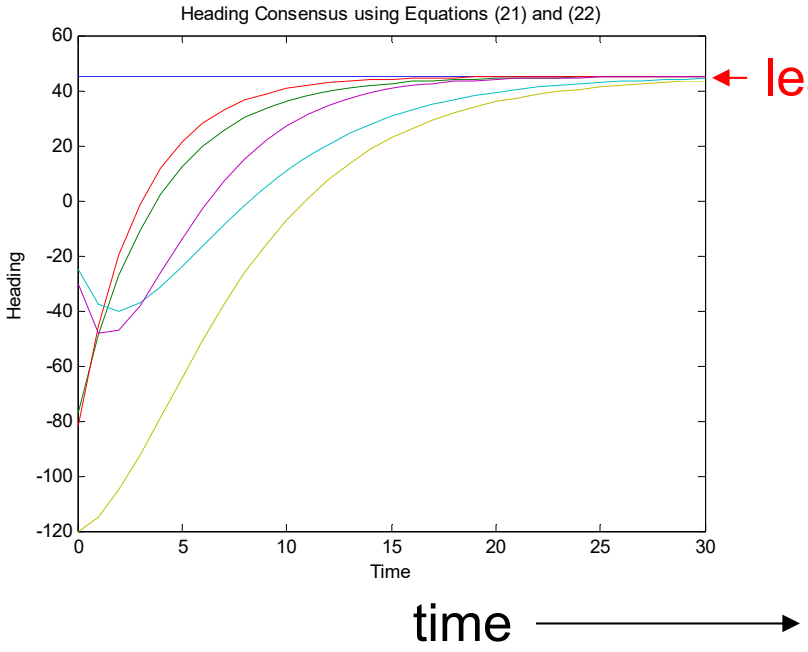


Formation- a Tree network

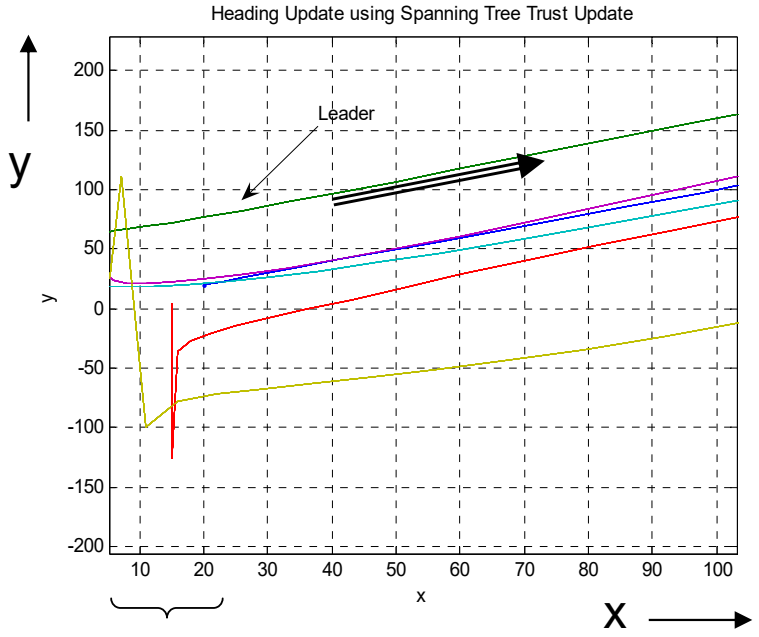
$$\dot{\theta}_i = \sum_{j \in N_i^c} a_{ij}(\theta_j - \theta_i)$$ heading angle

$$\dot{x}_i = V\cos\theta_i$$

$$\dot{y}_i = V\sin\theta_i$$

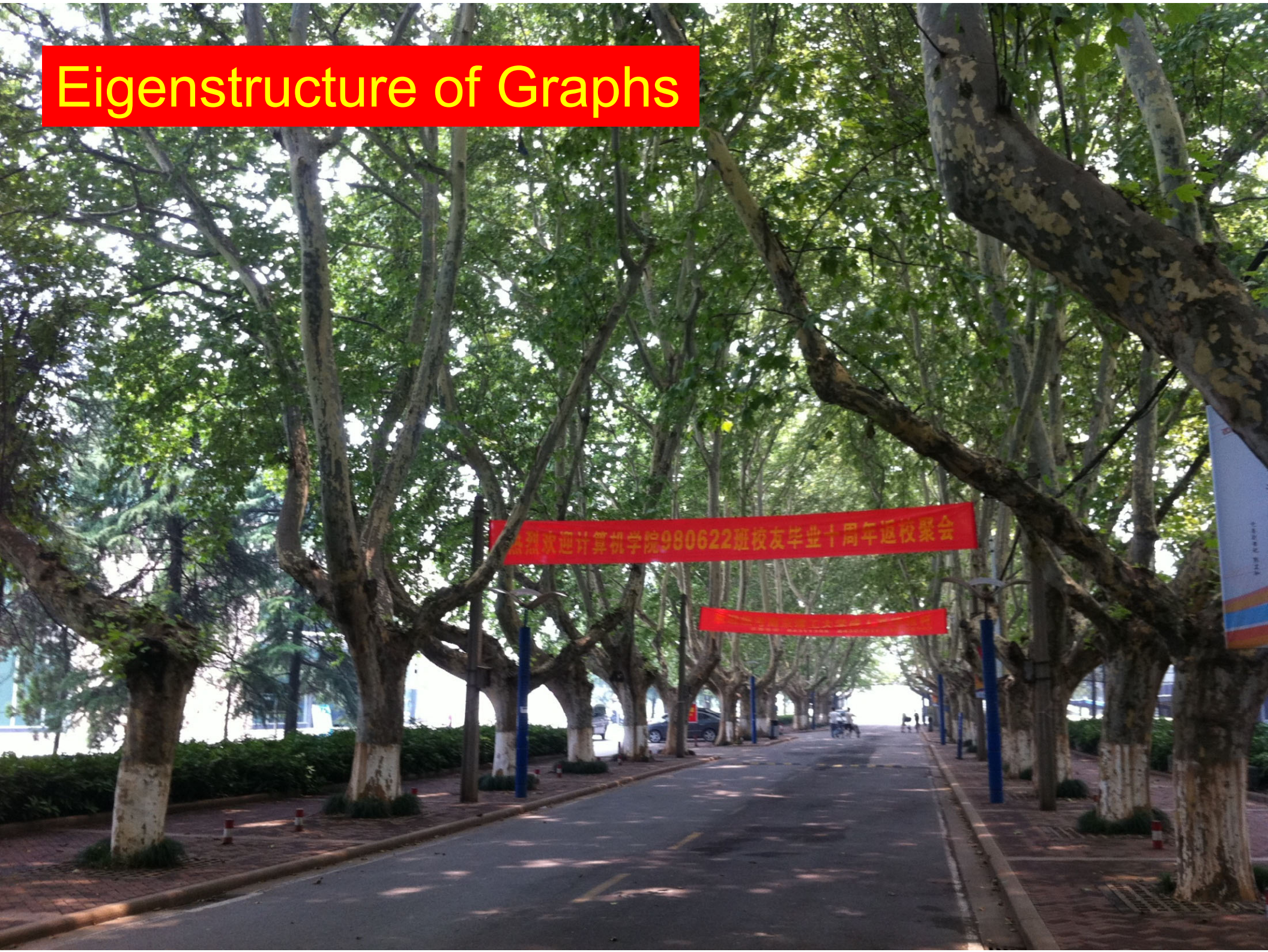


Convergence of headings



Nodes converge to heading of leader

Eigenstructure of Graphs



Closed-loop system with local voting protocol

$$\dot{x}_i = \sum_{j \in N_i} a_{ij}(x_j - x_i) \qquad \dot{x} = -Lx \qquad x(t) = e^{-Lt}x(0)$$

L= *D*-*A*= graph Laplacian matrix

Closed-loop dynamics depends on *L*

Eigenvalues of *L* determine time response

At steady-state $0 = \dot{x}_s = -Lx_s$

L =D-A has row sum of 0

$$0 = L\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} c = Lc\underline{1}$$

Therefore at steady state $x_s = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} c = c\underline{1}$ For any constant *c*

and consensus is reached

Eigenstructure of Graph Laplacian Matrix

State Space Transformation $L = MJM^{-1}$

In general the Jordan form matrix is $J = block\ diag \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$

Right Eigenvectors

We consider the case of simple L which has diagonal Jordan form

Then $J = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{bmatrix}$

Define transformation matrix as the modal matrix

$$M = \begin{bmatrix} v_1 & v_2 & \cdots & v_N \end{bmatrix}$$

So that

$$LM = MJ = L\begin{bmatrix} v_1 & v_2 & \cdots & v_N \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_N \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{bmatrix}$$

and $Lv_i = \lambda_i v_i$

or $(\lambda_i I - L)v_i = 0$

v_i is said to be a right eigenvector for the eigenvalue λ_i

Left Eigenvectors

$$L = MJM^{-1}$$

So can write either $LM = MJ$ as before

Or $M^{-1}L = JM^{-1}$

Define inverse modal matrix as $M^{-1} = W^T = \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_N^T \end{bmatrix}$

Then $M^{-1}L = JM^{-1}$ or $W^T L = JW^T$

$$M^{-1}L = JM^{-1} = \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_N^T \end{bmatrix} L = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{bmatrix} \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_N^T \end{bmatrix}$$

and $w_i^T L = \lambda_i w_i^T$

or $w_i^T (\lambda_i I - L) = 0$

w_i is said to be a left eigenvector for the eigenvalue λ_i

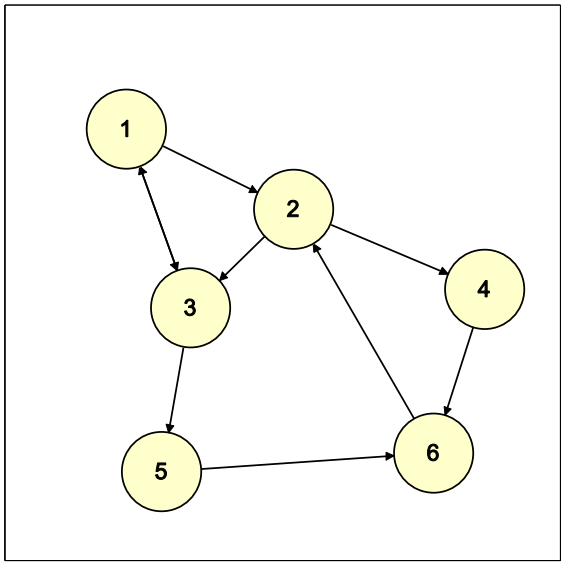
Eigenvalues in MATLAB

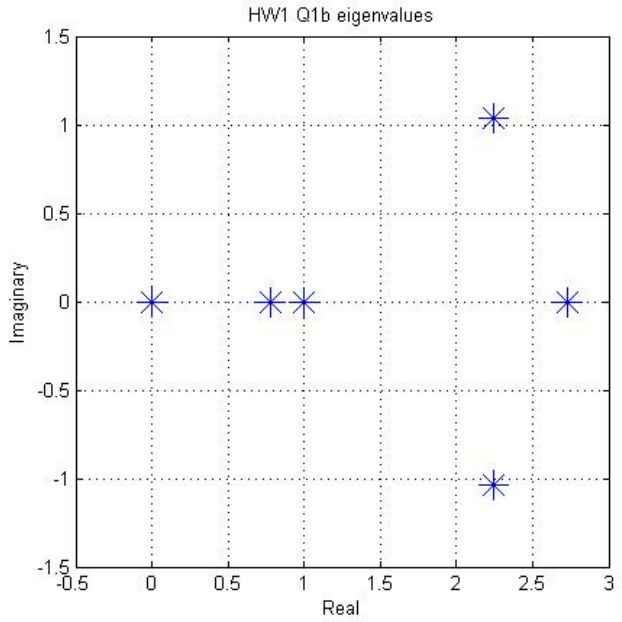
$$L = MJM^{-1}$$

[M,J,M^-T]= [M,J,W] = eig(L)

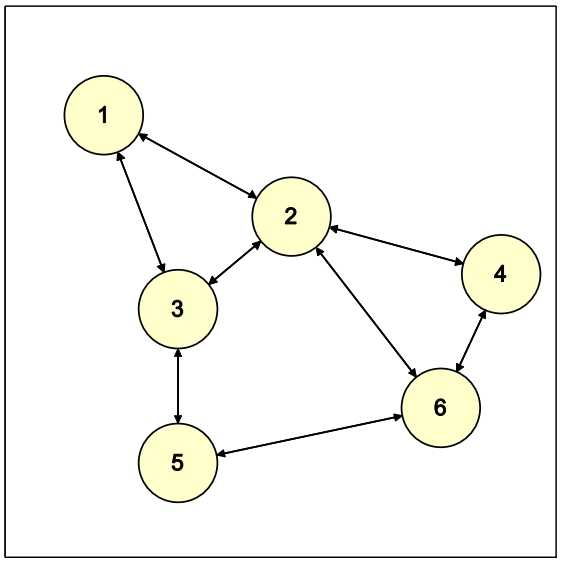
Graph Eigenvalues for Different Communication Topologies

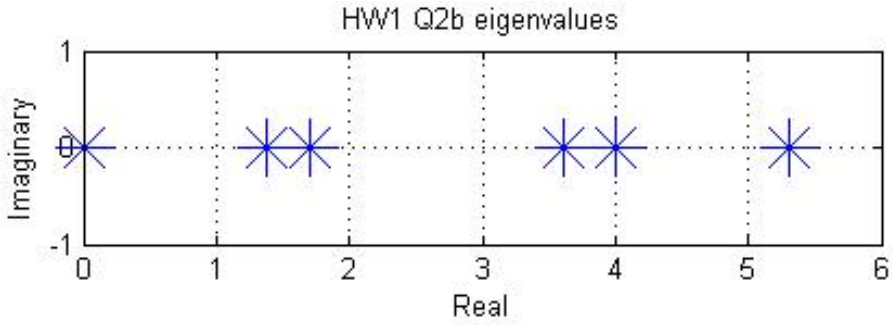
Directed graph-
Better conditioned





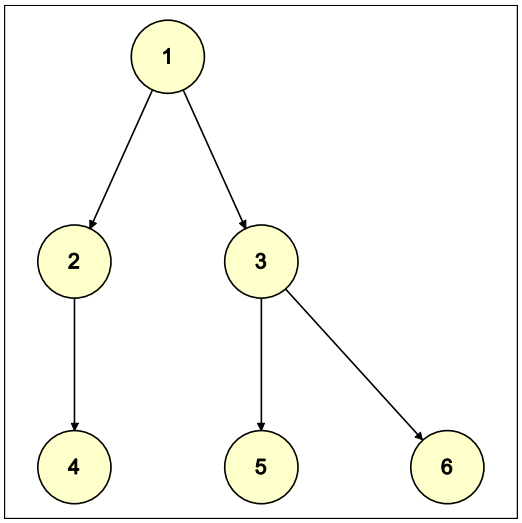
Undirected graph-
More ill-
conditioned

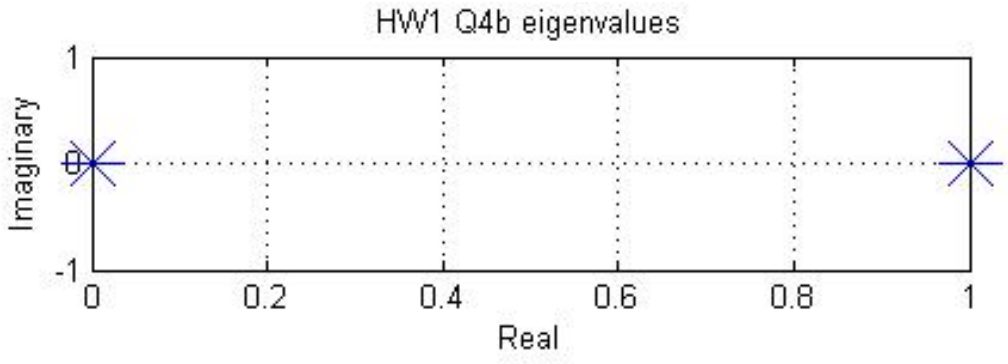




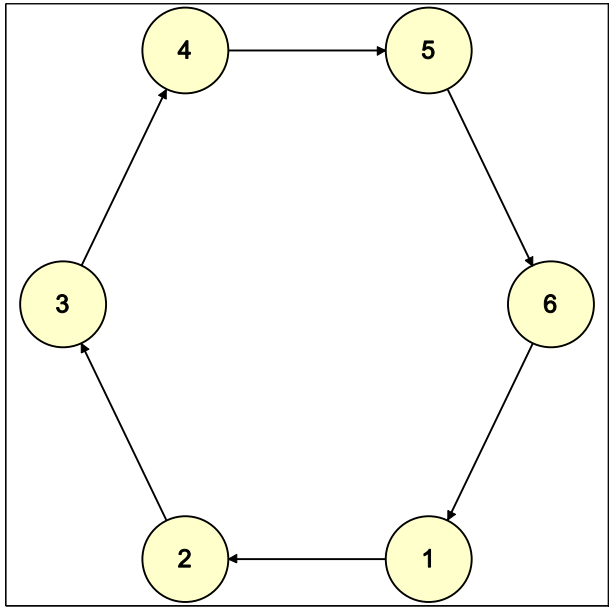
Graph Eigenvalues for Different Communication Topologies

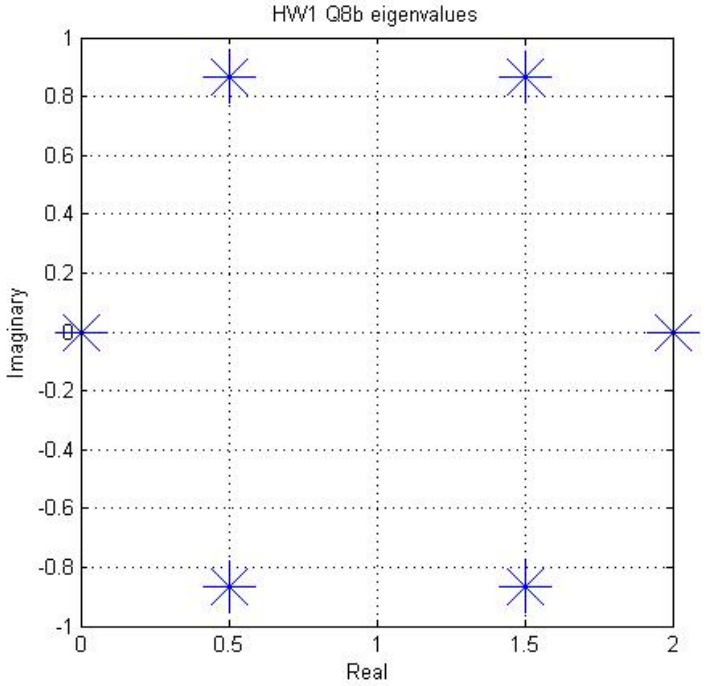
Directed Tree-
Chain of
command



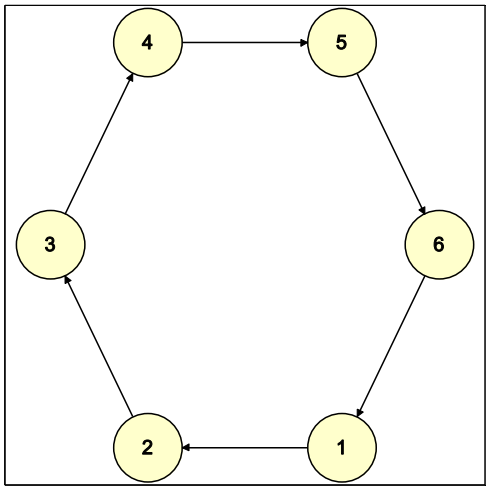


Directed Ring-
Gossip network
OSCILLATIONS

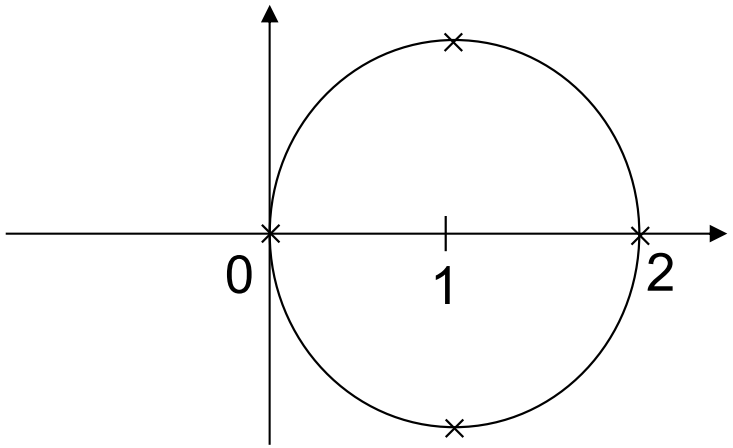




E.G. if graph is k-cyclic (all circular paths have a GCD of k)



then e-vals of $D^{-1}L$ are uniformly spaced around this circle

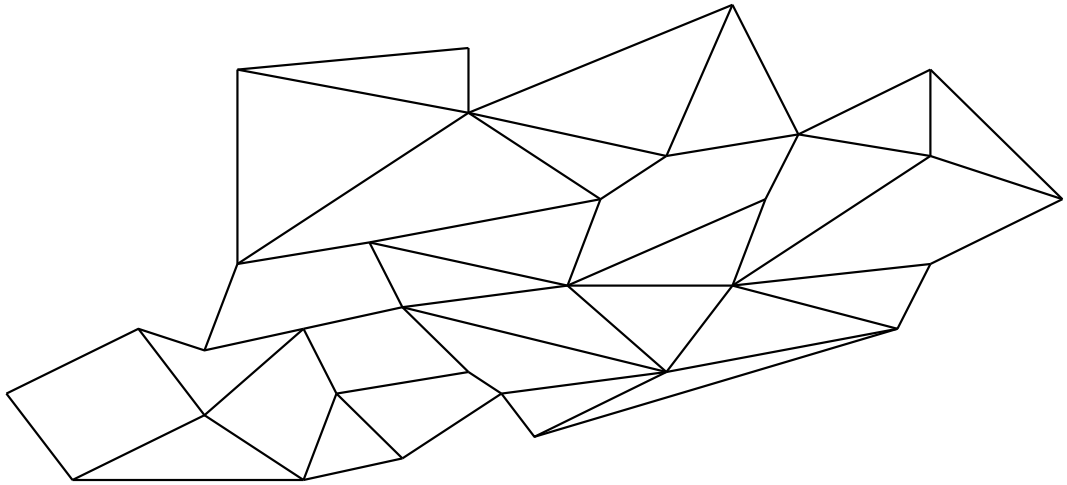


e.g. k=4

Then consensus time plot oscillates until steady-state

Captures the structure of social networks and rumor passing

Connectivity for Undirected Graphs



$$a_{ij} = a_{ji}$$

$$A = [a_{ij}] = [a_{ji}] = A^T$$

$$L = D - A = D - A^T = L^T$$

Connected if for all nodes *i* and *j* there is a path from *i* to *j*.
If there is a path from *i* to *j*, there is a path from *j* to *i*

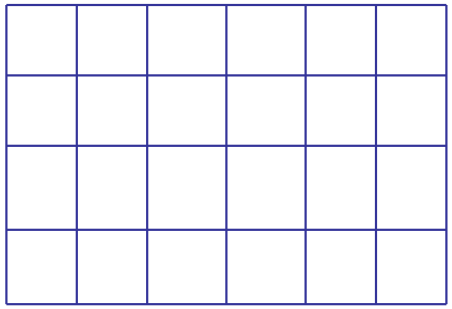
Any undirected graph has $L = L^T$

Hence, all its eigenvalues are real and can be ordered as

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$$

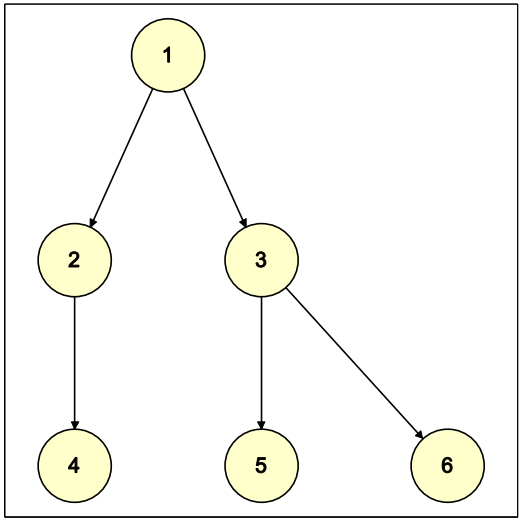
Synchronization on Good Graphs

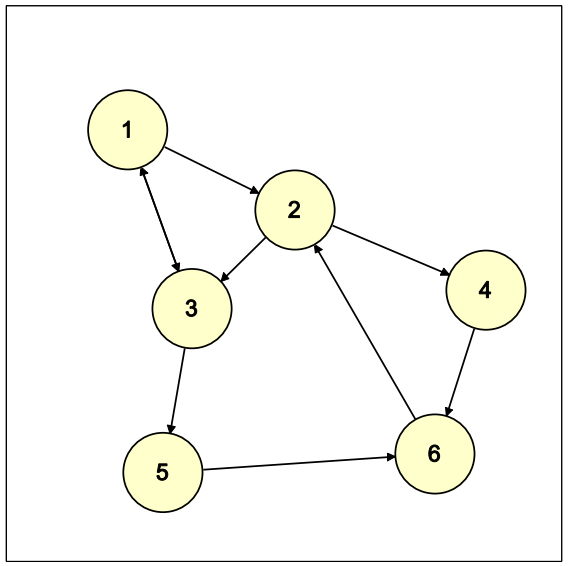
Chris Elliott fast video



Regular mesh

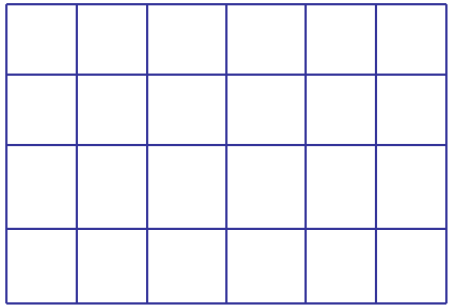
Eigenvalues of Graph Laplacian L



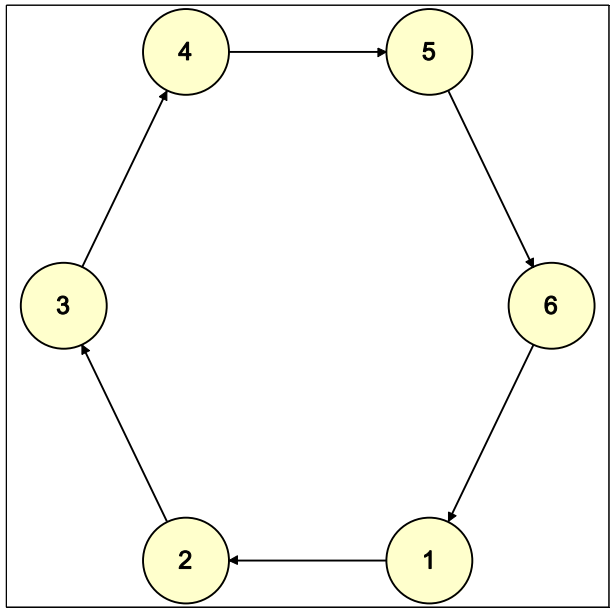


Synchronization on Gossip Rings

Chris Elliott weird video



Eigenvalues of Graph Laplacian L

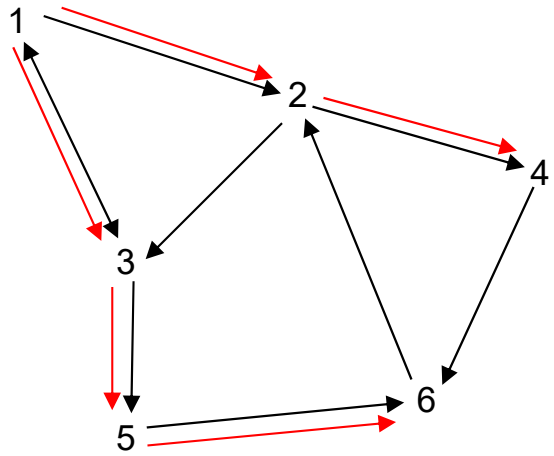


Since L has all row sums zero : $L\underline{1}c = 0$

where $\underline{1} = [1 \quad \cdots \quad 1]^T \in R^N$ and c is any constant

$0 = -L\underline{1}c = (\lambda_1 I - L)v_1$ if $\lambda_1 = 0$ and $v_1 = \underline{1}c$

so $\lambda_1 = 0$ Is an eigenvalue and $v_1 = \underline{1}c$ is the right e-vector



Theorem. Graph contains a spanning tree iff e-val at $\lambda_1 = 0$ is simple.

Then $\lambda_2 > 0$

Then *L* has one e-val at zero and all the rest in open right-half plane

Then *-L* has one e-val at zero and all the rest stable

Graph strongly connected implies exists a spanning tree

Theorem : L has rank N-1, i.e $\lambda_1 = 0$ is non-repeated, if and only if graph G has a spanning tree.

Modal decomposition

$$L = MJM^{-1} = \begin{bmatrix} v_1 & v_2 & \cdots & v_N \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{bmatrix} \begin{bmatrix} w_1^T \\ w_2^T \\ \vdots \\ w_N^T \end{bmatrix} = \sum_{j=1}^{N} v_i \lambda_i w_i^T$$

$$\dot{x} = -Lx$$

$$x(t) = e^{-Lt} x(0)$$

Modal decomposition

$$x(t) = e^{-Lt} x(0) = \sum_{j=1}^{N} v_i e^{-\lambda_i t} w_i^T x(0)$$

Closed-loop system with local voting protocol

$\dot{x} = -Lx$ Jordan Normal Form $L = MJM^{-1}$

Modal decomposition $x(t) = e^{-Lt}x(0) = \sum_{j=1}^{N} v_i e^{-\lambda_i t} w_i^T x(0)$

v_i = right e-vectors
w_i = left e-vectors

Closed-loop dynamics depends on eigenstructure of L

$$L = D - A = \begin{bmatrix} d_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & d_N \end{bmatrix} - \begin{bmatrix} 0 & a_{12} & \cdots & a_{1N} \\ a_{21} & 0 & & \vdots \\ \vdots & & \ddots & \\ a_{N1} & \cdots & & 0 \end{bmatrix} \quad , \quad d_i = \sum_{j=1}^{N} a_{ij} \text{ = row sum}$$

L has row sum of 0

$$0 = L\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = L\underline{1} \qquad \text{or} \qquad 0 = -L\underline{1} = (\lambda_1 I - L)v_1$$

so $\lambda_1 = 0$ and $v_1 = \underline{1}$ is the right e-vector

Consensus: Final Consensus Value

Closed-loop system with local voting protocol

$$\dot{x} = -Lx$$

Modal decomposition $x(t) = e^{-Lt}x(0) = \sum_{j=1}^{N} v_i e^{-\lambda_i t} w_i^T x(0) = \sum_{j=1}^{N} \left(w_i^T x(0) \right) e^{-\lambda_i t} v_i$

Let $\lambda_1 = 0$ be simple. Then at steady-state

$$x(t) \to v_1 e^{-\lambda_1 t} w_1^T x(0) = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} \gamma_1 & \cdots & \gamma_N \end{bmatrix} \begin{bmatrix} x_1(0) \\ \vdots \\ x_N(0) \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \sum_{j=1}^{N} \gamma_j x_j(0)$$

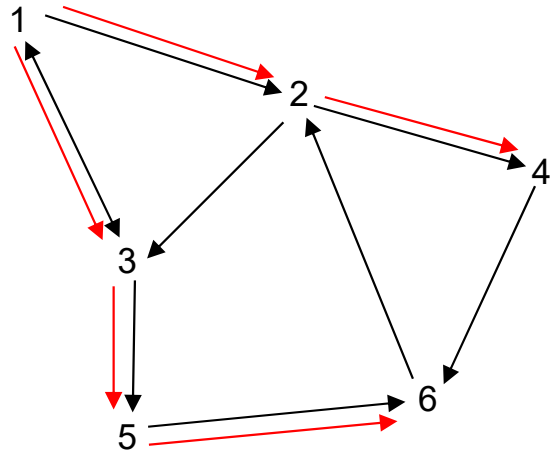
with $w_1^T = \begin{bmatrix} \gamma_1 & \cdots & \gamma_N \end{bmatrix}$ the normalized left e-vector of $\lambda_1 = 0$

Therefore $x_i(t) \to \sum_{j=1}^{N} \gamma_j x_j(0)$ for all nodes CONSENSUS

Consensus value depends on communication graph structure

Importance of left e-vector of $\lambda_1 = 0$

Consensus Leaders



Spanning tree
Root node

CONSENSUS $x_i(t) \to \sum_{j=1}^{N} \gamma_j x_j(0)$ for all nodes

with $w_1^T = [\gamma_1 \quad \cdots \quad \gamma_N]$ the normalized left e-vector of $\lambda_1 = 0$

Theorem. $\gamma_i \neq 0$ iff node i is a Root node

So the consensus value is a weighted average of the initial conditions of all the Root nodes

Corollary. Let the graph be a tree. Then all nodes converge to IC of the root node

Convergence Rate

Closed-loop system with local voting protocol

$$\dot{x} = -Lx$$

Modal decomposition $x(t) = e^{-Lt}x(0) = \sum_{j=1}^{N} v_i e^{-\lambda_i t} w_i^T x(0) = \sum_{j=1}^{N} \left(w_i^T x(0) \right) e^{-\lambda_i t} v_i$

Let $\lambda_1 = 0$ be simple. Then for large t

$$x(t) \to v_2 e^{-\lambda_2 t} w_2^T x(0) + v_1 e^{-\lambda_1 t} w_1^T x(0) = v_2 e^{-\lambda_2 t} w_2^T x(0) + \underline{1} \sum_{j=1}^{N} \gamma_j x_j(0)$$

λ_2 determines the rate of convergence and is called the FIEDLER e-value

There is a big push to find expressions for the left e-vector for $\lambda_1 = 0$ and the Fiedler e-val λ_2

Consensus value is

$$x_i(t) \to \sum_{j=1}^{N} \gamma_j x_j(0)$$ = weighted average of initial conditions of nodes

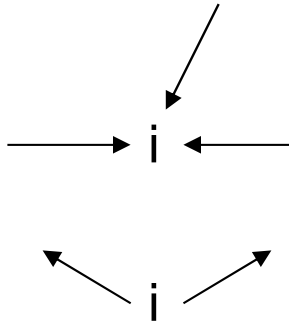
Consensus value depends on communication graph structure

A graph is balanced if in-degree=out-degree

$$A = [a_{ij}]$$

$$d_i = \sum_{j=1}^{N} a_{ij}$$ Row sum= in-degree

$$d_i^o = \sum_{i=1}^{N} a_{ij}$$ Column sum= out-degree



Balanced means that row sum= column sum

Then L has row sum=0 and column sum=0

$w_1^T L = 0$ means that left e-vector is also $w_1 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \underline{1}$

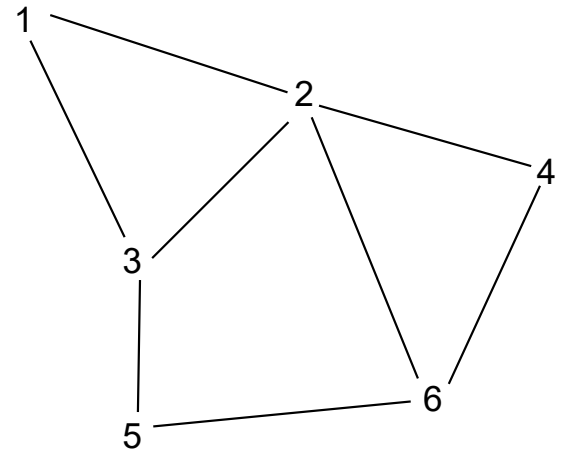
Then Consensus value is

$$x_i(t) \to \frac{1}{N} \sum_{j=1}^{N} x_j(0)$$ = average of initial conditions of nodes

Independent of graph structure

Undirected Graphs

e_{ij} is an edge if e_{ji} is an edge, and $e_{ij} = e_{ji}$



$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} = A^T$$

A is symmetric

Row sum= column sum so that in-degree= out-degree

L=D-A is symmetric and positive semidefinite

A symmetric graph is balanced

Connected undirected graph has average consensus value

Then Consensus value is

$$x_i(t) \to \frac{1}{N}\sum_{j=1}^{N} x_j(0)$$ = average of initial conditions of nodes

Independent of graph structure

Graph Laplacian Eigenstructure

Laplacian $L = D - A$ First order consensus system $\dot{x} = -Lx$

First eigenvalue is $\lambda_1 = 0$ and $v_1 = \underline{1}c$ is the right e-vector

All other eigenvalues have positive real parts
Iff there exists a spanning tree
A sufficient condition for this is strongly connected.

Then the Fiedler eigenvalue λ_2 has positive real part
And is a measure of the convergence rate

What about the other eigenvalues? Where are they?

Thm. Gerschgorin Circle Criterion. All eigenvalues of matrix $E = [e_{ij}] \in R^{n \times n}$

are located within the following union of n discs

$$\bigcup_{i=1}^{n} \left\{ z \in C : |z - e_{ii}| \le \sum_{j \neq i} |e_{ij}| \right\}$$

$$L = D - A = \begin{bmatrix} d_1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & d_N \end{bmatrix} - \begin{bmatrix} 0 & a_{12} & \cdots & a_{1N} \\ a_{21} & 0 & & \vdots \\ \vdots & & \ddots & \\ a_{N1} & \cdots & & 0 \end{bmatrix}$$

All e-vals are in the circles $C(d_i, $

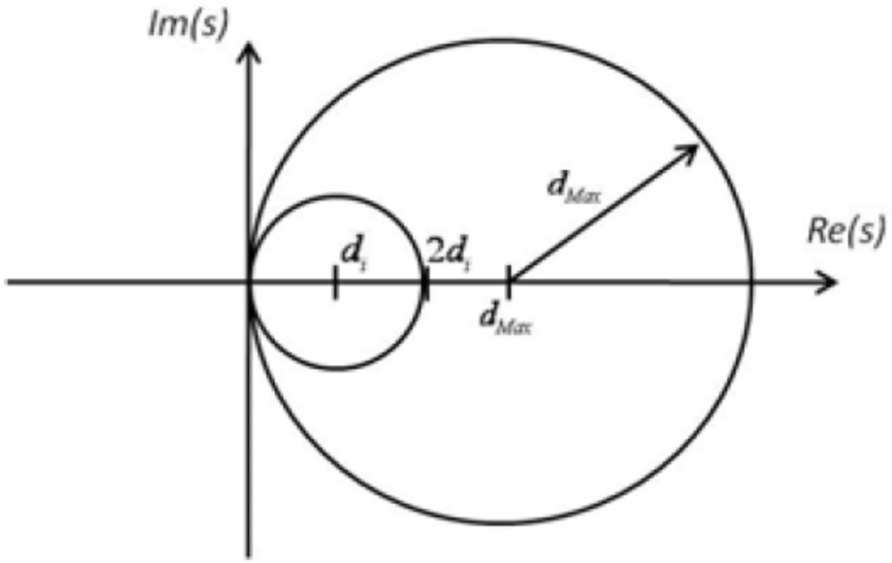


Fig. 2.3. Geršgorin discs of L in the complex plane.

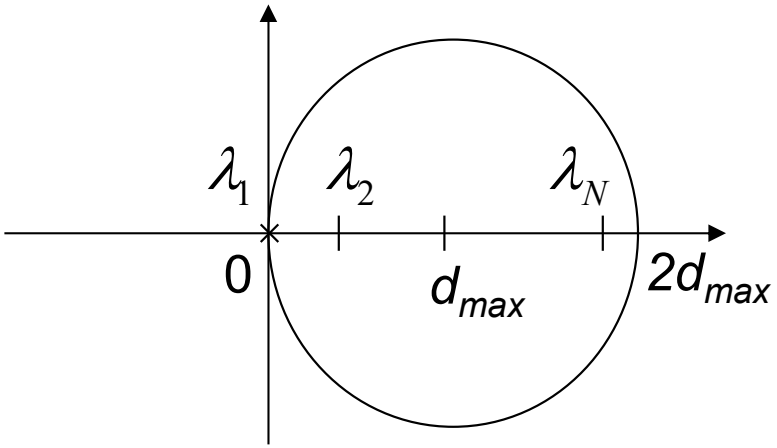
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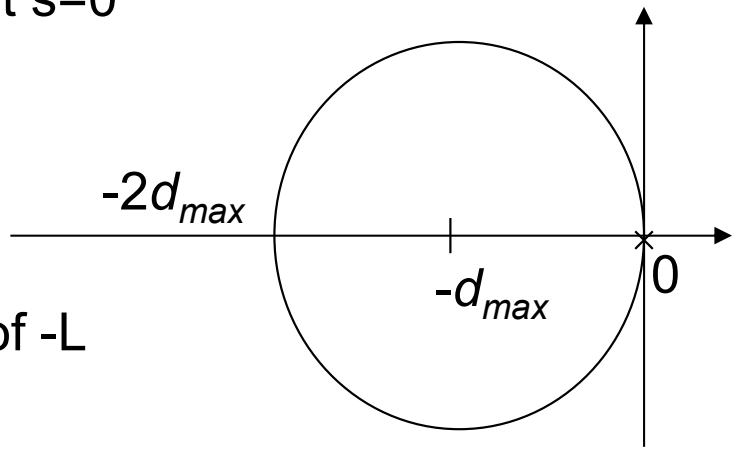
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All e-vals are in the circles $C(d_i, d_i)$



L has row sum = 0 implies there is an eigenvalue at s=0

$$0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$$



E-vals of -L

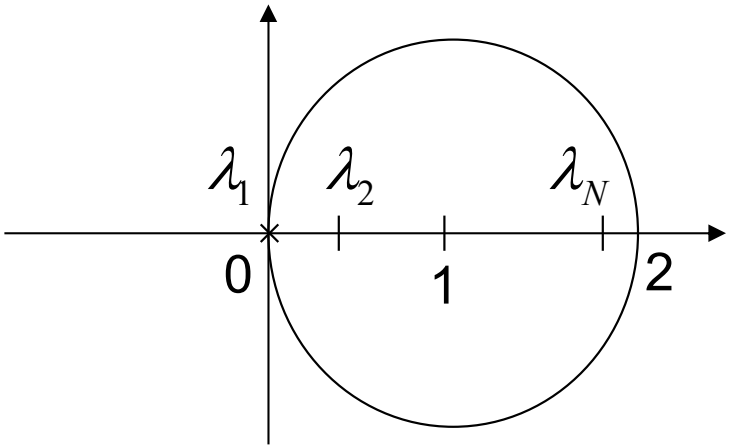
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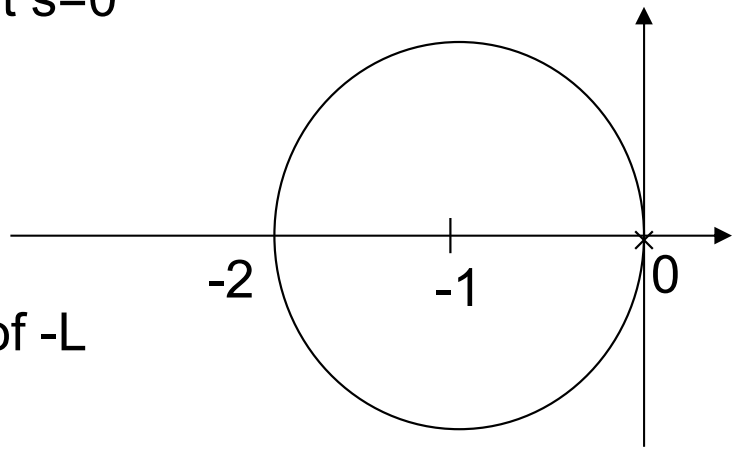
Normalized Laplacian $D^{-1}L = D^{-1}(D - A) = I - D^{-1}A$ has diagonal entries =1

All e-vals are in the circle



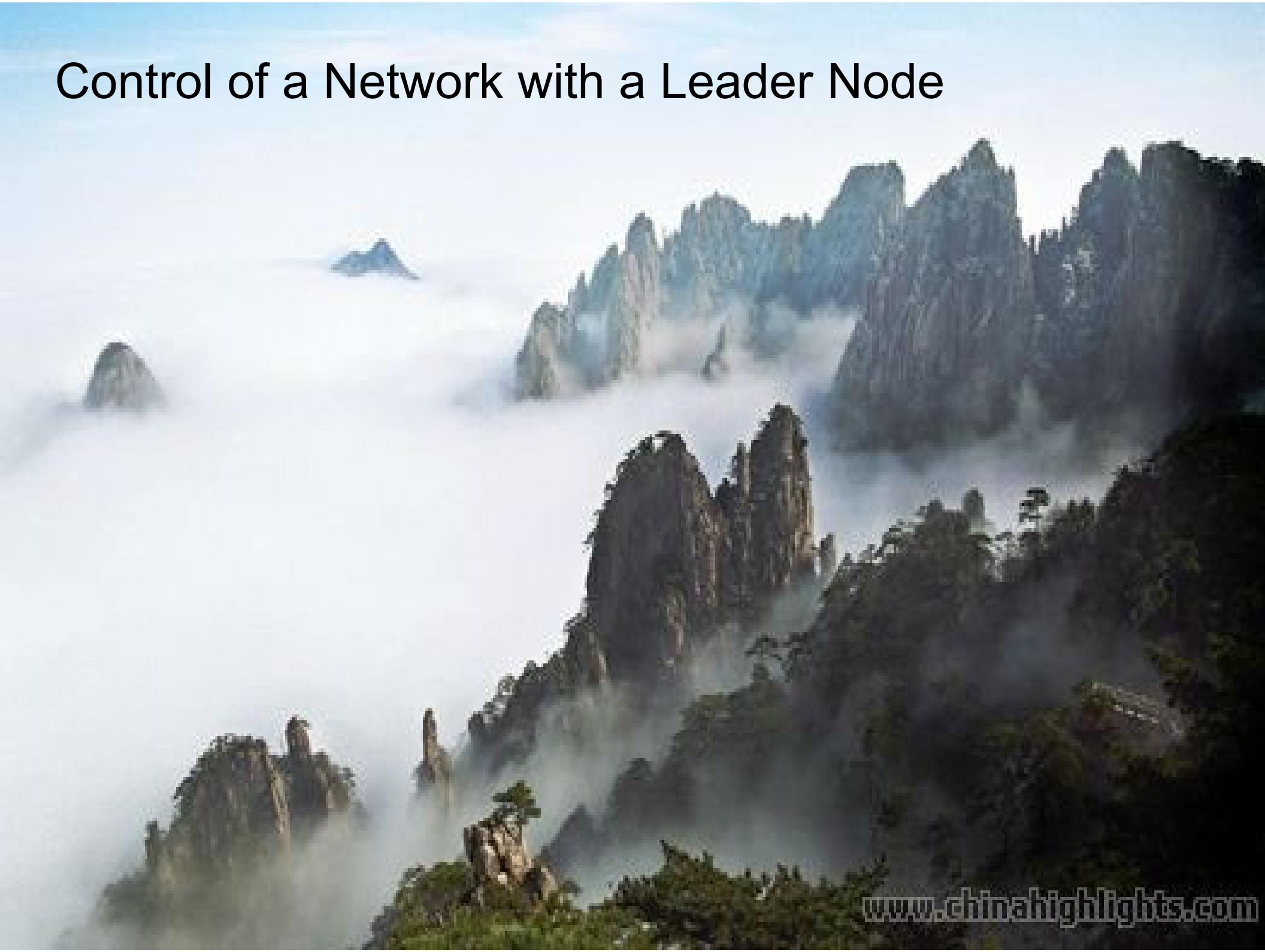
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E-vals of -L

Control of a Network with a Leader Node



Controlled Consensus: Cooperative Tracker

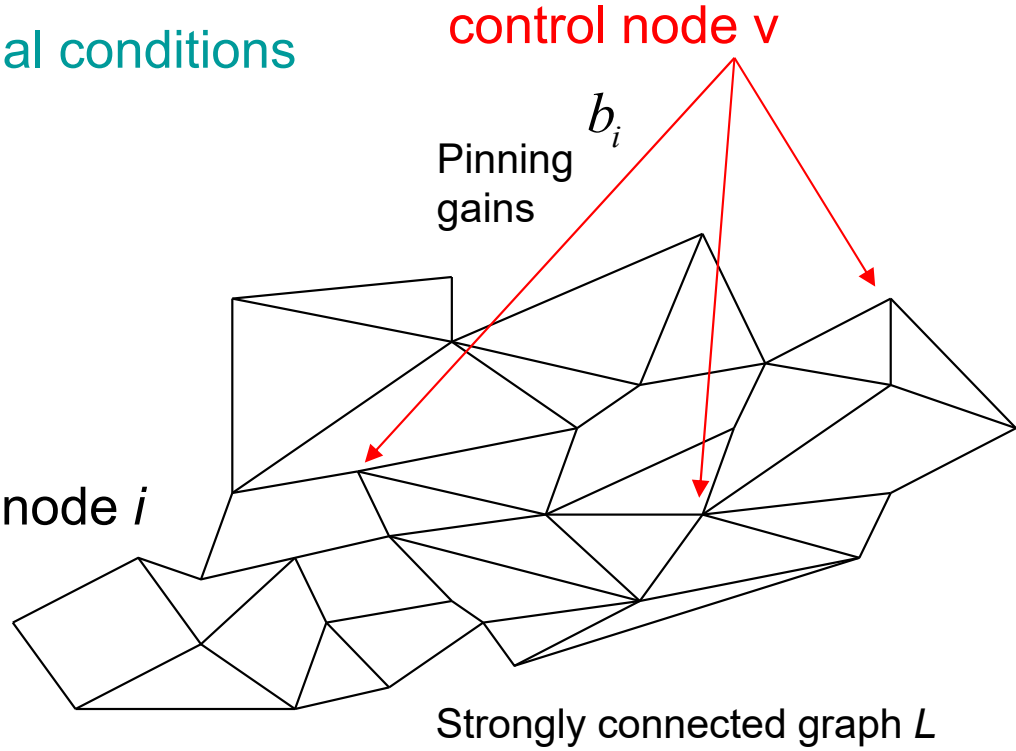
Ron Chen

Get rid of dependence on initial conditions

Node state $\dot{x}_i = u_i$

Local voting protocol with control node *v*

$$u_i = \sum_{j \in N_i} a_{ij}(x_j - x_i) \quad + b_i(v - x_i)$$



$b_i \neq 0$ If control v is in the neighborhood of node *i*

$$u_i = -\left(b_i + \sum_{j \in \bar{N}_i} a_{ij} \right) x_i + \sum_{j \in N_i} a_{ij} x_j \quad + b_i v$$

Control node is in some neighborhoods $\bar{N}_i$

Pinning Control

$$\dot{x} = -(L + B)x + B\underline{1}v \qquad B = diag\{b_i\}$$

Theorem. Let at least one $b_i \neq 0$. Then L+B is nonsingular with all e-vals positive and -(L+B) is asymptotically stable

So initial conditions of nodes in graph A go away.
Consensus value depends only on v

In fact, v is now the only spanning node

Controlled Consensus: Cooperative Tracker

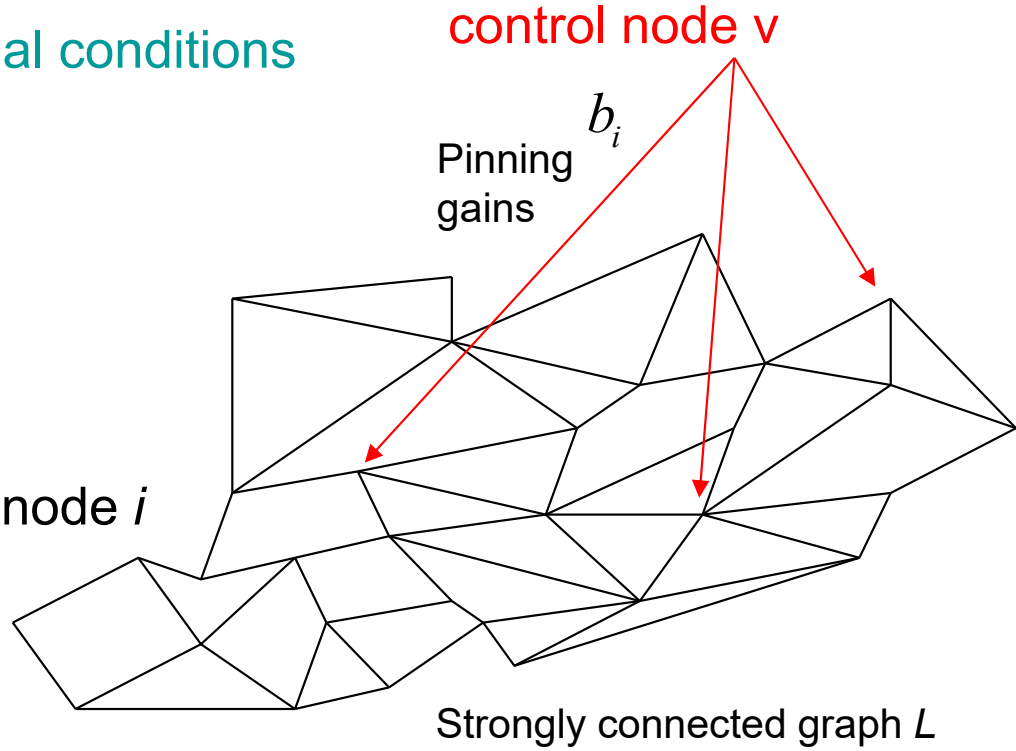
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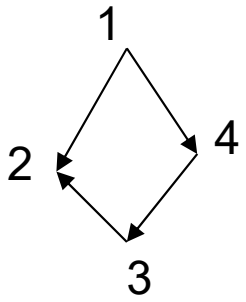
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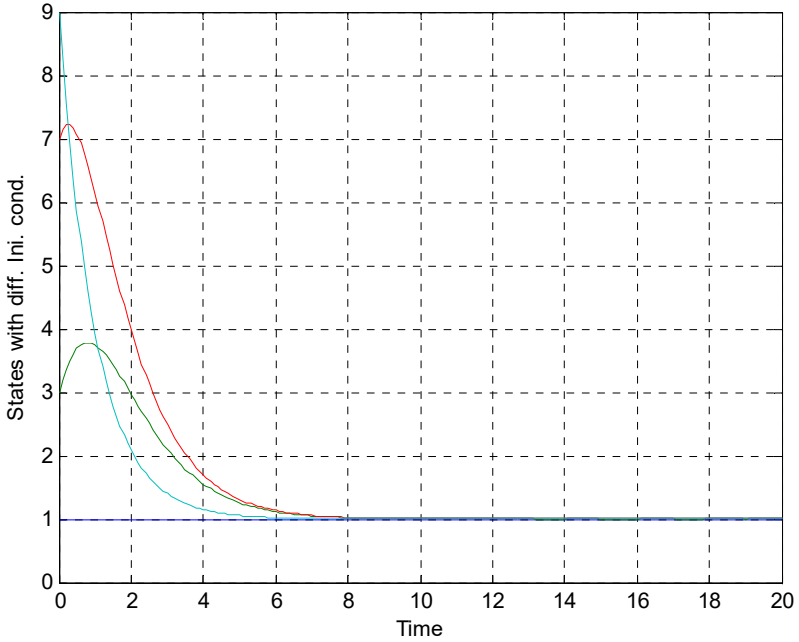
Controlled Consensus

Original network



A = [0 0 0 0;
1 0 1 0;
0 0 0 1;
1 0 0 0];

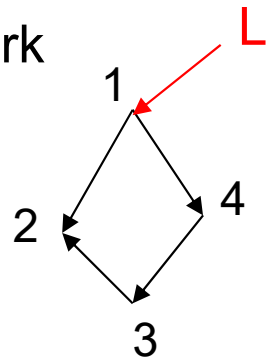
Lamdas = 0 1 1 2



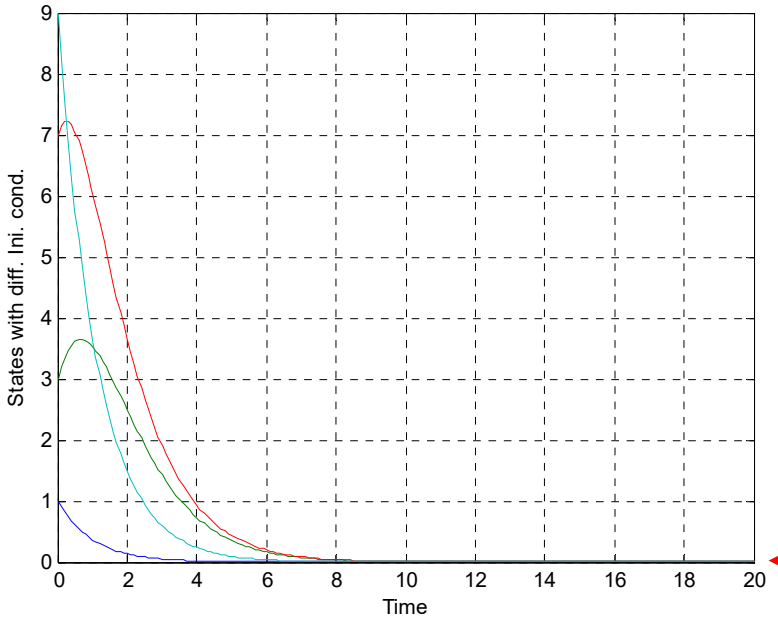
Average of ICS

Consensus time approx 7.5 sec

Controlled network



Lamdas = 0 1 1 1 2



Leader's IC

Consensus time approx 8 sec

