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1. Cooperative Tracker: Controlled Synchronization

Equation Chapter 1 Section 1
Add discussion. Refer to refs.

1.1 Cooperative Tracker for Single-Integrator Dynamics

Consider a communication digraph $Gr = (V, E)$ of N agents, with the allowed information flow between the agents indicated by the edge arrows of the graph. Endow each agent with the scalar single-integrator dynamics

$$\dot{x}_i = u_i \tag{1.1.1}$$

with $x_i, u_i \in R$. It is desired for the states of all the nodes to synchronize to the state $x_0(t)$ of a leader or control node which has the, generally nonautonomous, that is time-varying, dynamics

$$\dot{x}_0 = f(x_0, t) \tag{1.1.2}$$

A special case is $\dot{x}_0 = 0$, which means that all nodes should synchronize to the same constant consensus value $x_0(0)$. The drift term $f(x_0, t)$ could represent, e.g. motion dynamics of the control node. That is, we assume that the control node can have a time-varying state. Since all agents should synchronize to a given reference state so that $x_i(t) \to x_0(t), \forall i$, this is a cooperative tracker problem. The leader dynamics (1.1.2) can be considered as a command generator system, or exo-system, that generates the desired trajectory for the group.

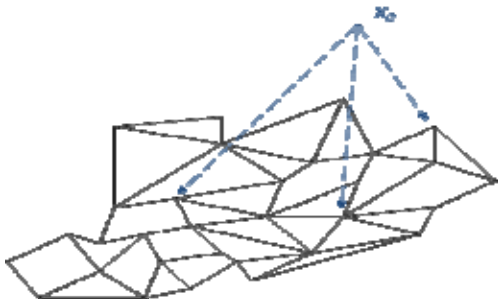


Figure 1. A control node pins into a few nodes in a graph to control the actions of the entire group.

It is desired to achieve synchronization to the state of the leader when only a small percentage of nodes have direct access to information about the leader. This is in the spirit of [Couzin], where it was found that a large group of animals can achieve heading and speed consensus even though only a few individuals are informed about the desired direction and speed of motion. This scenario is shown in Figure *.

Protocol for Leader Following and Error Dynamics

Consider the local protocol control law

$$u_i = c \sum_{j \in N_i} a_{ij}(x_j - x_i) \quad + \quad c g_i(x_0 - x_i) \tag{1.1.3}$$

with a_{ij} the graph edge weights and c a control gain. This uses only information from the neighbor set N_i of node i (the set of nodes with edges incoming into node i). The gains $g_i \geq 0$

are nonzero only for a small percentage of nodes i that are directly connected by edges to the leader. That is, $g_i > 0$ if and only if there exist an edge from the control node to the i-th node in *Gr*. This has been called pinning control [Ron Chen], with $g_i > 0$ the pinning gains. It is assumed that $g_i > 0$ for at least one i. We refer to the nodes i for which $g_i > 0$ as the pinned or controlled nodes.

Define the global state vector $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_N \end{bmatrix}^T \in R^N$ and the global disagreement vector [Olfati-Saber and Murray 2004]

$$\delta = \left(x - \underline{x}_0 \right) \in R^N \tag{1.1.4}$$

where $\underline{x}_0 = \underline{1} x_0 \in R^N$, with $\underline{1}$ the N-vector of ones. Note that the elements of δ are $\delta_i(t) = x_i(t) - x_0(t)$, so synchronization is achieved if and only if $\delta_i(t) \to 0, \forall i$.

Synchronization can be analyzed by examining the *local neighborhood synchronization error* defined as [Khoo, Xie, Man 2009]

$$e_i = \sum_{j \in N_i} a_{ij} \left(x_j - x_i \right) + g_i \left(x_0 - x_i \right) \tag{1.1.5}$$

for each node i. This captures all the information available to node i for controls purposes. Write

$$e_i = -\sum_{j \in N_i} a_{ij} x_i + \sum_{j \in N_i} a_{ij} x_j - g_i x_i + g_i x_0 = -g_i x_i - d_i x_i + \sum_{j \in N_i} a_{ij} x_j + g_i x_0 \tag{1.1.6}$$

with $d_i = \sum_j a_{ij}$ the in-degree of node i. Define the global neighborhood error vector $e = \begin{bmatrix} e_1 & e_2 & \cdots & e_N \end{bmatrix}^T \in R^N$ so that this is equivalent to

$$e = -Gx - Dx + Ax + G\underline{1} x_0 = -Gx - Lx + G\underline{1} x_0 \tag{1.1.7}$$

with $A = [a_{ij}]$ the graph adjacency matrix, $D = diag(d_i)$ the diagonal matrix of in-degrees, $G = diag(g_i)$ the diagonal matrix of pinning gains, and $L = D - A$ the graph Laplacian matrix. Since the row sums of L are zero, $L\underline{1} x_0 = 0$ and this can be written as

$$e = -\left(L + G \right) \left(x - \underline{1} x_0 \right) = -\left(L + G \right) \left(x - \underline{x}_0 \right) = -\left(L + G \right) \delta \tag{1.1.8}$$

Lemma 1. Let at least one pinning gain g_i be nonzero and the graph *Gr* be strongly connected. Then $(L + G)$ is nonsingular and has all eigenvalues strictly in the right-half plane.

Proof: Strong connectivity implies that L is irreducible (Chapter on matrices *). If at least one g_i is nonzero, then $(L + G)$ is irreducibly diagonally dominant and hence nonsingular. Gershgorin's Circle Criterion shows that all eigenvalues of $(L + G)$ are in the right-half plane. Since $(L + G)$ is nonsingular, they are all strictly in the right-half plane. ■

Given a matrix M, denote its minimum singular value (Appendix A) as $\underline{\sigma}(M)$ and its maximum singular value as $\bar{\sigma}(M)$. If M is nonsingular one has

$$\bar{\sigma}(M^{-1}) = \frac{1}{\underline{\sigma}(M)} \tag{1.1.9}$$

The Sylvester inequality states that for any square matrix M and compatibly dimensioned vector y

$$\underline{\sigma}(M)\|y\|^2 \le y^T My \le \bar{\sigma}(M)\|y\|^2 \tag{1.1.10}$$

The maximum singular value $\bar{\sigma}(M)$ is the spectral norm or induced 2-norm of M. As such, it satisfies the Cauchy-Schwarz inequality, for any compatibly dimensioned matrices M, N

$$\bar{\sigma}(MN) \le \bar{\sigma}(M)\bar{\sigma}(N) \tag{1.1.11}$$

The spectral norm is compatible with the vector 2-norm, so that for any matrix M and compatibly dimensioned vector y, one has by Cauchy-Schwarz

$$\|My\|_2 \le \bar{\sigma}(M)\|y\|_2 \tag{1.1.12}$$

Subsequently, we omit the subscript '2' on the norm symbol for notational convenience.

Lemma 2. Relation Between Neighborhood Error and Disagreement Vector. Let at least one pinning gain g_i be nonzero and the graph Gr be strongly connected. Then

$$\|\delta\| \le \|e\| / \underline{\sigma}(L+G) \tag{1.1.13}$$

Moreover, then the nodes synchronize to the state of the leader node if and only if the local neighborhood tracking errors $e_i(t)$ are zero for all i.

Proof: Under the hypotheses $(L+G)$ is nonsingular. Therefore, according to (1.1.8)

$$\delta = -(L+G)^{-1} e \tag{1.1.14}$$

so that (1.1.13) follows by the Cauchy-Schwarz inequality. Clearly $e = 0$ if and only if $\delta = 0$.

■

In fact, a necessary and sufficient condition for the singularity of $(L+G)$ is existence of a spanning tree, with the pinning gain nonzero for the root node of the spanning tree. This means that synchronization can be guaranteed in a tree formation, for instance, if the control node pins into the root node of the tree. Therefore, all the following results hold if the graph is not strongly connected but only has a spanning tree and the pinning goes to the root node.

Note that the control inputs are given by

$$u_i = ce_i \tag{1.1.15}$$

so that the global control input vector $u = [u_1 \cdots u_N]^T \in R^N$ is given by

$$u = ce = -c(L+G)(x - \underline{x}_0) = -c(L+G)\delta \tag{1.1.16}$$

This is tracking controller with a feedback term from $x(t)$ and a feedforward term from the leader's state $\underline{x}_0(t)$. The global closed-loop dynamics are

$$\dot{x} = ce = -c(L+G)(x - \underline{x}_0) \tag{1.1.17}$$

The global disagreement dynamics are

$$\dot{\delta} = (\dot{x} - \dot{\underline{x}}_0) = ce - \underline{f}(x_0,t) = -c(L+G)\delta - \underline{f}(x_0,t) \tag{1.1.18}$$

where $\underline{f}(x_0,t) = \underline{1} f(x_0,t)$. The global neighborhood error dynamics are

$$\dot{e} = -(L+G)(\dot{x} - \underline{\dot{x}}_0) = -c(L+G)e + (L+G)\,\underline{f}(x_0, t) \tag{1.1.19}$$

According to Lemma 1, both error systems are asymptotically stable with a disturbance input given by the leader node dynamics $\underline{f}(x_0, t)$.

Non-Scalar Node States

It was assumed that the node states and controls in the dynamics (1.1.1) are scalars $x_i, u_i \in R$. If the states and controls are vectors so that $x_i, u_i \in R^n$, then $x = [x_1^T \cdots x_N^T]^T \in R^{nN}$, $u = [u_1^T \cdots u_N^T]^T \in R^{nN}$ and the elements a_{ij}, d_i, and g_i in (1.1.6) are multiplied by the $n \times n$ identity matrix I_n. Then, the global dynamics are written as

$$u = -c((L+G) \otimes I_n)(x - \underline{x}_0) \tag{1.1.20}$$

$$\dot{x} = -c((L+G) \otimes I_n)(x - \underline{x}_0) \tag{1.1.21}$$

with $\otimes$ the Kronecker product (Appendix A). The following developments still hold if the Kronecker product is added where appropriate.

For motion in the plane, one would of course select $x_i = [p_i(t)\ \ q_i(t)]^T \in R^2$ where $(p_i(t), q_i(t))$ is the position of node *i* in the (x, y) plane.

Simulation of Pinning Control for Nonscalar States

For simulation, you can avoid the Kronecker product by defining the *Nxn* matrix of states as

$$X = \begin{bmatrix} x_1^T \\ x_2^T \\ \\ x_N^T \end{bmatrix}$$

Then

$$\dot{X} = -c(L+G)X + c(L+G)\underline{1}x_0^T$$

Lyapunov Stability Analysis of Cooperative Tracker

It is desired now to analyze the protocols (1.1.3) to determine when they lead to synchronization of all agents to the leader's state. This can be achieved by analyzing the stability of the neighborhood error dynamics (1.1.19). A Lyapunov analysis will be used (Appendix A), to which the next two results are the key. They show two alternative methods of constructing a diagonal solution *P* to a Lyapunov equation in $(L+G)$. Note that if *L* is irreducible and at least one pinning gain is nonzero, then $(L+G)$ is a nonsingular M Matrix (Chapter *), that is, all off diagonal elements are nonpositive and all leading minors are nonnegative.

Lemma 3. Cooperative Control Lyapunov equation. Let $(L+G)$ be a nonsingular M matrix. Define

$$p = \begin{bmatrix} p_1 & p_2 & \cdots & p_N \end{bmatrix}^T = \left(L+G\right)^{-T} \underline{1} \tag{1.1.22}$$

$$P = \text{diag}\{p_i\} \tag{1.1.23}$$

Then $P > 0$ and the matrix Q defined as

$$Q = P\left(L+G\right) + \left(L+G\right)^T P \tag{1.1.24}$$

is positive definite.

Proof: [H.W. Zhang and F.L. Lewis, Automatica, 2015]. ■

Lemma 4. Cooperative Control Lyapunov equation. Let $(L+G)$ be a nonsingular M matrix. Then there exist $p, q \succ 0$ such that

$$(L+G)q \succ 0, \quad p^T(L+G) \succ 0 \tag{1.1.25}$$

where denotes $\succ$ that all elements are positive. Define the diagonal matrix

$$P = \text{diag}\{p_i / q_i\} \tag{1.1.26}$$

Then $P > 0$ and the matrix Q defined as

$$Q = P\left(L+G\right) + \left(L+G\right)^T P \tag{1.1.27}$$

is positive definite.

Proof: [Qu 2009]. ■

The following assumption is required.

Assumption 1. The leader dynamics $f(x_0,t)$ is unknown to the agents but uniformly bounded so that $\|f(x_0,t)\| \le F_M, \forall t$.

This assumption means that the velocity of the leader node is unknown but bounded above. Although the bound F_M is assumed to exist, it is not used in the control design and does not have to be known.

The next result show when protocol (1.1.3) induces synchronization to the trajectory of the control node. The next definition provides a notion of 'practical' stability, so that the synchronization is achieved 'closely enough' for practical purposes.

Definition 1. Uniform Ultimate Boundedness (UUB). A time signal $y(t) \in R^n$ is said to be uniformly ultimately bounded if there exists a compact set $S \subset R^n$ such that for all $y(0) \in S$ there exists a bound B and a time $T(y(0), B)$ such that $\|y(t)\| < B$ for all times $t > T$.

Theorem 1. Cooperative Tracking. Let at least one pinning gain g_i be nonzero, the graph Gr be strongly connected, and Assumption 1 hold. Then under the protocols (1.1.3) the local neighborhood errors are UUB and converge to the residual set defined by

$$B = \frac{2}{c}\sqrt{N}F_M \frac{\bar{\sigma}(P)\bar{\sigma}(L+G)}{\underline{\sigma}(Q)} \tag{1.1.28}$$

with P, Q defined in Lemma 3 or Lemma 4. The synchronization error norm $\|e(t)\|$ convergences for large initial errors $\|e(0)\|$ to this set exponentially according to

$$\|e(t)\| \le \beta \|e(0)\| e^{-\alpha t} \tag{1.1.29}$$

with $\beta = \sqrt{\bar{\sigma}(P) / \underline{\sigma}(P)}$ and convergence rate

$$\alpha = \frac{c}{2} \frac{\underline{\sigma}(Q)}{\bar{\sigma}(P)} \tag{1.1.30}$$

Remark. This theorem shows that the tracking error can be decreased, and the speed of synchronization can be increased, by increasing the control gains c in protocol (1.1.3).

Proof: Consider the Lyapunov function candidate define in terms of the neighborhood tracking error as

$$V = \frac{1}{2} e^T P e \tag{1.1.31}$$

with $0 < P = P^T \in R^{N \times N}$ the diagonal matrix defined in Lemma 3 or Lemma 4. Then from (1.1.19)

$$\dot{V} = -e^T P \dot{e} = -e^T P (L+G) \left[ce - \underline{f}(x_0, t) \right] = -ce^T P (L+G) e + e^T P (L+G) \underline{f}(x_0, t)$$

Since L is irreducible and G has at least one diagonal entry $g_i > 0$, then $(L+G)$ is a nonsingular M matrix. Defining therefore Q according to Lemma 3 or Lemma 4 one has

$$\dot{V} = -\frac{1}{2} ce^T Q e + e^T P (L+G) \underline{f}(x_0, t)$$

Note that $\|\underline{1} f(x_0, t)\| = \sqrt{N} \| f(x_0, t) \|$. Therefore, from Sylvester's inequality and Cauchy-Schwarz

$$\dot{V} \le -\tfrac{1}{2} c \, \underline{\sigma}(Q) \|e\|^2 + \sqrt{N} F_M \bar{\sigma}(P) \bar{\sigma}(L+G) \|e\|$$

This is negative if $\|e(t)\| > B$ with B defined in (1.1.28). Therefore, the steady-state errors are practically bounded inside the ball with radius defined by (1.1.28). That is, if $\|e(t)\|$ goes outside the ball of radius B, then the Lyapunov function decreases and $\|e(t)\|$ decreases. This means that $\|e(t)\|$ is practically bounded by radius B.

One has for large $\|e(t)\|$, approximately

$$V = \tfrac{1}{2} e^T P e$$

$$\dot{V} \le -\tfrac{1}{2} c \, \underline{\sigma}(Q) \|e\|^2$$

Now Sylvester's inequality

$$\tfrac{1}{2} \underline{\sigma}(P) \|e\|^2 \le V = \tfrac{1}{2} e^T P e \le \tfrac{1}{2} \bar{\sigma}(P) \|e\|^2 \tag{1.1.32}$$

reveals that

$$\dot{V} \le -\tfrac{1}{2} c\, \underline{\sigma}(Q) \|e\|^2 \le -c \frac{\underline{\sigma}(Q)}{\bar{\sigma}(P)} V = -2\alpha V$$

with α given by (1.1.30) Therefore $V(t) \le V(0) e^{-2\alpha t}$. Using (1.1.32) again on $V(0)$ and $\|e(0)\|$ yields the result (1.1.29). ■

According to this theorem and Lemma 2, the global disagreement error is practically bounded within the ball of radius

$$\frac{2}{c} F_M \frac{\bar{\sigma}(P)\bar{\sigma}(L+G)}{\underline{\sigma}(Q)\underline{\sigma}(L+G)} \tag{1.1.33}$$

Lyapunov Design for Distributed Control Laws

It is important to note that the design given here relies on the fact that proper selection of the Lyapunov function in Theorem 1 results in a tracking control law that is distributed and can be implemented locally at each node using only information from its neighbor nodes. The key is the selection of a diagonal matrix P in the Lyapunov function (1.1.31) that satisfies a Lyapunov equation (1.1.24) involving the matrix $(L+G)$. Moreover, it is important to use the local neighborhood error $e(t)$ in the Lyapunov function, not the disagreement vector $\delta(t)$ which would not yield a local protocol.

1.2 Cooperative Tracker for Double-Integrator Dynamics

Equation Section (Next)

Consider a digraph $Gr = (V, E)$ of N agents, each having the second-order Newton's Law node dynamics $\ddot{x}_i = u_i$ or

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= u_i \end{aligned} \tag{1.2.1}$$

with positions, velocities, and control inputs $x_i, v_i, u_i \in R$. It is desired for the states of all the nodes to synchronize to the state of a leader or control node which has the reference trajectory generator dynamics

$$\begin{aligned} \dot{x}_0 &= v_0 \\ \dot{v}_0 &= u_0 \end{aligned} \tag{1.2.2}$$

This means that all agents should synchronize their positions and velocities to the leader.

It is assumed for convenience of notation that the node positions, velocities, and controls are scalars. If they are vectors so that $x_i, v_i, u_i \in R^n$, then the following development still holds if the Kronecker product (Appendix A) $((L+G) \otimes I_n)$ is used instead of $(L+G)$. For motion in the plane, one would select $x_i = [p_i(t) \;\; q_i(t)]^T \in R^2$ where $(p_i(t), q_i(t))$ is the position of node i in the (x, y) plane, and similarly for the velocity and input.

Error Dynamics

Define the position and velocity local neighborhood tracking errors

$$e_i = \sum_{j \in N_i} a_{ij}(x_j - \Delta_j - x_i + \Delta_i) + g_i(x_0 - x_i + \Delta_i) \tag{1.2.3}$$

$$e_{2i} = \sum_{j \in N_i} a_{ij}(v_j - v_i) + g_i(v_0 - v_i) \tag{1.2.4}$$

with Δ_i the constant desired position offset in the formation of node *i* from the leader node. These capture all the information available to node *i* for controls purposes. The pinning gains g_i are only nonzero for a small subset of the nodes in the graph.

Define global vectors $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_N \end{bmatrix}^T \in R^N$, $v = \begin{bmatrix} v_1 & v_2 & \cdots & v_N \end{bmatrix}^T \in R^N$, $u = \begin{bmatrix} u_1 & u_2 & \cdots & u_N \end{bmatrix}^T \in R^N$. Then

$$\dot{x} = v, \quad \dot{v} = u \tag{1.2.5}$$

Define $e = \begin{bmatrix} e_1 & e_2 & \cdots & e_N \end{bmatrix}^T \in R^N$, $e_2 = \begin{bmatrix} e_{21} & e_{22} & \cdots & e_{2N} \end{bmatrix}^T \in R^N$, $\Delta = \begin{bmatrix} \Delta_1 & \Delta_2 & \cdots & \Delta_N \end{bmatrix}^T \in R^N$. Then

$$e = -(L+G)(x - \Delta) + G\underline{1}x_0 = -(L+G)(x - \Delta - \underline{1}x_0) \tag{1.2.6}$$

$$e_2 = -(L+G)v + G\underline{1}v_0 = -(L+G)(v - \underline{1}v_0) \tag{1.2.7}$$

where one has used the fact that $L\underline{1} = 0$.

According to Lemma 1, if at least one pinning gain g_i is nonzero and the graph *Gr* is strongly connected, then $(L+G)$ is nonsingular and has all eigenvalues strictly in the right-half plane. The next result follows therefore readily.

Lemma 4. Let at least one pinning gain g_i be nonzero and the graph *Gr* be strongly connected. Then

$$\left\| (x - \Delta) - \underline{1}x_0 \right\| \le \left\| e \right\| / \underline{\sigma}(L+G) \tag{1.2.8}$$

$$\left\| v - v_0 \right\| = \left\| e_2 \right\| / \underline{\sigma}(L+G) \tag{1.2.9}$$

Moreover, then the nodes synchronize to the state of the leader node so that $x_i(t) = x_0(t) + \Delta_i$, $v_i(t) = v_0(t), \forall i$ if and only if the local neighborhood position and velocity tracking errors $e_i(t), e_{2i}(t)$ are zero for all *i*. ■

A necessary and sufficient condition for the singularity of $(L+G)$ is existence of a spanning tree, with the pinning gain nonzero for the root node of the spanning tree. Therefore, all the results here hold if the graph is not strongly connected but only has a spanning tree and the pinning goes to the root node.

Now we can design a cooperative tracker for second-order synchronization using the local neighborhood errors. Define therefore the local neighborhood sliding mode errors

$$s_i(t) = e_{2i}(t) + \alpha e_i(t) \tag{1.2.10}$$

with $\alpha > 0$. In terms of global graph vectors

$$s(t) = e_2(t) + \alpha e(t) \tag{1.2.11}$$

One can write the global error dynamics as

$$\dot{e} = -(L+G)(\dot{x} - \underline{1}\dot{x}_0) = -(L+G)(v - \underline{1}v_0) = e_2 \tag{1.2.12}$$

$$\dot{e}_2 = -(L+G)(\dot{v} - \underline{1}\dot{v}_0) = -(L+G)(u - \underline{1}u_0) \tag{1.2.13}$$

$$\dot{s} = \dot{e}_2 + \alpha\dot{e} = -(L+G)(u - \underline{1}u_0) + \alpha e_2 \tag{1.2.14}$$

Define the sliding variables

$$r_i(t) = v_i(t) + \alpha x_i(t)$$

$$r(t) = v(t) + \alpha x(t)$$

Then from (1.2.11), (1.2.6), (1.2.7) one has

$$s = -(L+G)(v - \underline{1}v_o) - \alpha(L+G)(x - \Delta - \underline{1}x_0) = -(L+G)(r - \underline{1}r_0 - \alpha\Delta)$$

so that

$$\dot{s} = -(L+G)(\dot{r} - \underline{1}\dot{r}_0) = -(L+G)(u + \alpha v - \underline{1}\dot{r}_0) \tag{1.2.15}$$

Protocol and Lyapunov Analysis for Second-Order Cooperative Tracking

We shall now provide a Lyapunov analysis to select an appropriate protocol that guarantees cooperative tracking for the second-order systems. Since these ideas are important in understanding the idiosyncrasies of cooperative control, a didactic development of the proof is provided. The following assumption is required.

Assumption 2. The leader acceleration $u_0(t)$ and velocity $v_0(t)$ are uniformly bounded. Hence, the leader's variable $\dot{r}_0(t) = u_0(t) + \alpha v_0(t)$, which is unknown to the agents, is uniformly bounded so that $\|\dot{r}_0\| \le F_M$, $\forall t$.

Although the bound F_M is assumed to exist, it is not used in the control design and does not have to be known.

To achieve tracking synchronization select the Lyapunov function candidate

$$V = \tfrac{1}{2}s^T P s + \tfrac{1}{2}e^T e \tag{1.2.16}$$

with P the diagonal matrix in Lemma 3 or Lemma 4. Then

$$\dot{V} = s^T P\dot{s} + e^T\dot{e} = -s^T P(L+G)(u + \alpha v - \underline{1}\dot{r}_0) + e^T e_2$$

Select now the cooperative tracking protocol

$$u = cs - \alpha v \tag{1.2.17}$$

or

$$u_i = cs_i - \alpha v_i = c(e_{2i}(t) + \alpha e_i(t)) - \alpha v_i \tag{1.2.18}$$

with control gain $c > 0$. This is a distributed protocol given in terms of the neighbors' positions and velocities in terms of the local neighborhood errors (1.2.3), (1.2.4). Then

$$\begin{aligned}\dot{V} &= s^T P\dot{s} + e^T\dot{e} = -cs^T P(L+G)s + s^T P(L+G)\underline{1}\dot{r}_0 + e^T e_2 \\ &= -cs^T P(L+G)s + s^T PG\underline{1}\dot{r}_0 + e^T(s-\alpha e) \\ &= -\tfrac{1}{2}cs^T Qs - \alpha e^T e + s^T PG\underline{1}\dot{r}_0 + s^T e\end{aligned}$$

where $Q>0$ is defined by Lyapunov equation (1.1.24). Using now Sylvester and Cauchy-Schwarz one writes

$$\dot{V} \le -\tfrac{1}{2}cq\|s\|^2 - \alpha\|e\|^2 + b\|s\| + \|s\|\cdot\|e\|$$

where $q \equiv \underline{\sigma}(Q)$ and $b = \sqrt{N}F_M\bar{\sigma}(P)\bar{\sigma}(G)$. (Note that $\|\underline{1}\dot{r}_0\| = \sqrt{N}\,\|\dot{r}_0\|$.) One can write this as

$$\dot{V} \le -\frac{1}{2}\begin{bmatrix}\|e\| \\ \|s\|\end{bmatrix}^T\begin{bmatrix}2\alpha & 1 \\ 1 & cq\end{bmatrix}\begin{bmatrix}\|e\| \\ \|s\|\end{bmatrix} + \begin{bmatrix}0 & b\end{bmatrix}\begin{bmatrix}\|e\| \\ \|s\|\end{bmatrix} \equiv -\frac{1}{2}z^T Mz + m^T z \le -\tfrac{1}{2}\underline{\sigma}(M)\|z\|^2 + \|m\|\|z\|$$

Matrix *M* is positive definite if the following conditions hold

$$\begin{aligned}&\alpha > 0 \\ &\alpha c > \frac{1}{2\underline{\sigma}(Q)}\end{aligned} \tag{1.2.19}$$

Then, $\underline{\sigma}(M) > 0$.

The Lyapunov derivative is negative, $\dot{V} < 0$, if

$$\|z\| > 2\frac{\|m\|}{\underline{\sigma}(M)} = 2\frac{\sqrt{N}F_M\bar{\sigma}(P)\bar{\sigma}(G)}{\underline{\sigma}(M)} \equiv B$$

Sufficient conditions for this are

$$\|e\| > 2\frac{\sqrt{N}F_M\bar{\sigma}(P)\bar{\sigma}(G)}{\underline{\sigma}(M)} \equiv B \tag{1.2.20}$$

or

$$\|s\| > 2\frac{\sqrt{N}F_M\bar{\sigma}(P)\bar{\sigma}(G)}{\underline{\sigma}(M)} \equiv B \tag{1.2.21}$$

Therefore, if $\|e(t)\|$ or $\|s(t)\|$ go outside the ball of radius *B*, then the Lyapunov function decreases and these norms decrease. This means that these norms are practically bounded to the ball of radius *B*.

We have just shown that, if the graph has a spanning tree with at least one pinning gain nonzero, into the root node, then the sliding errors $s(t)$ and the position errors $e(t)$ are UUB with bound *B*. In view of (1.2.11), the velocity errors $e_2(t)$ are also UUB. Thus, synchronization to the position and velocity of the control node is achieved to within some errors. One notes that the errors can be decreased by increasing the control gain *c*, for larger *c* yields larger $\underline{\sigma}(M)$. Note further that $\underline{\sigma}(M)$ depends on $q = \underline{\sigma}(Q)$.

1.3 Synchronization of Nonlinear Dynamical Systems

Equation Section (Next)

Synchronization phenomena are manifested throughout nature, physics, and chemistry and have fascinating properties. Some of these phenomena are the synchronization of the frequencies of illumination of fireflies, chirps of cicadas, and coupled oscillators, In this section we will examine some of these ideas by examining synchronization of coupled nonlinear dynamical systems. The idea here are from [Ron Chen]

Protocol for Coupled Nonlinear Systems

Consider N systems coupled on a communication digraph $Gr = (V, E)$ with dynamics

$$\begin{aligned} \dot{x}_i &= f(x_i) + u_i \\ y_i &= Cx_i \end{aligned} \tag{1.3.1}$$

where $x_i, u_i \in R^n$, the nonlinearities $f(.)$ at each node are identical, and $y_i(t) \in R^p$ are measured outputs. It is desired for all agents to synchronize to the leader's dynamics given by the command generator

$$\begin{aligned} \dot{x}_o &= f(x_0) \\ y_0 &= Cx_0 \end{aligned} \tag{1.3.2}$$

Select the control protocols

$$u_i = c\sum_j a_{ij}(y_j - y_i) + cd\, g_i(y_0 - y_i) = c\sum_j a_{ij}C(x_j - x_i) + cd\, g_i C(x_0 - x_i) \tag{1.3.3}$$

with control gains $c, d > 0$ pinning gains $g_i \geq 0$, and only a small percentage of the agents having $g_i > 0$. Define the global state vector $x = [x_1^T \cdots x_N^T]^T \in R^{nN}$ and global control vector $u = [u_1^T \cdots u_N^T]^T \in R^{nN}$. Then

$$\begin{aligned} u &= -c\big((D + dG - A) \otimes C\big)x + c\big(dG \otimes C\big)\underline{I}x_0 \\ &= -c\big((L + dG) \otimes C\big)x + c\big(dG \otimes C\big)\underline{x}_0 \end{aligned} \tag{1.3.4}$$

with $A = [a_{ij}]$ the graph adjacency matrix, $D = diag(d_i)$ the diagonal matrix of in-degrees, $G = diag(g_i)$ the diagonal matrix of pinning gains, and $L = D - A$ the graph Laplacian matrix. Vector $\underline{x}_0 = \underline{I}x_0 \equiv (\underline{1} \otimes I_n)x_0 \in R^{nN}$, with $\underline{1} \in R^n$ the vector of ones and I_n the $n \times n$ identity matrix. Since the row sums of L are zero, $L\underline{1} = 0$ and this can be written as

$$u = -c\big((L + dG) \otimes C\big)(x - \underline{x}_0) \tag{1.3.5}$$

The closed-loop dynamics are

$$\dot{x} = f(x) - c\big((L + dG) \otimes C\big)(x - \underline{x}_0) \tag{1.3.6}$$

Define errors

$$e_i = x_i - x_0 \tag{1.3.7}$$

Then the error dynamics are

$$\dot{e}_i = \dot{x}_i - \dot{x}_0 = f(x_i) - f(x_0) + u_i \tag{1.3.8}$$

Write this in terms of global N-vectors as

$$\dot{e} = f(x) - \underline{f}(x_0) - c\big((L+dG)\otimes C\big)(x - \underline{x}_0) \tag{1.3.9}$$

with $\underline{f}(x_0) = \underline{I}\ f(x_0) \in R^{nN}$, or

$$\dot{e} = \tilde{f}(x) - c\big((L+dG)\otimes C\big)e \tag{1.3.10}$$

with $\tilde{f}(x) = f(x) - \underline{f}(x_0) \in R^{nN}$.

Analysis of Synchronization Stability

To study when the protocol (1.3.3) achieves synchronization of the coupled systems, let us propose the Lyapunov function

$$V = \tfrac{1}{2} e^T P e \tag{1.3.11}$$

With P the diagonal matrix defined in Lemma 3 or Lemma 4. Then

$$\dot{V} = e^T P \dot{e} = e^T P\Big(\tilde{f}(x) - c(L+dG)\otimes Ce\Big)$$

$$\dot{V} = e^T P\tilde{f}(x) - ce^T P(L+dG)\otimes Ce \tag{1.3.12}$$

$$\dot{V} = e^T P\tilde{f}(x) - \tfrac{1}{2} ce^T Qe \tag{1.3.13}$$

Matrix Q is defined by

$$P\big(-(L+dG)\otimes C\big) + \big(-(L+dG)\otimes C\big)^T P = -Q \tag{1.3.14}$$

and is positive definite if . since $-(L+dG)\otimes C$ is asymptotically stable if C has positive eigenvalues. Note that the eigenvalues of $M\otimes N$ are $\lambda_i(M)\mu_j(N)$

Case 1: Linearize the dynamics.

Linearize the dynamics $f_i(x)$ about the leader node nonlinearity $f(x_0)$. Then linearized system is

$$\dot{e} = \big(\nabla\underline{f}(x_0) - c(L+dG)\otimes C\big)e \equiv Ee$$

and

$$\dot{V} = e^T P\dot{e} = e^T P\big(\nabla\underline{f}(x_0) - c(L+d\bar{B})\otimes\Gamma\big)e$$

$$= e^T PEe = \tfrac{1}{2} e^T(PE + E^T P)e = -\tfrac{1}{2} e^T Qe$$

if E is AS.

Case 2. Dynamics is Lipschitz

Then

$$\big\|f(x) - \underline{f}(x_0)\big\| \le \kappa\big\|x - \underline{x}_0\big\|$$

and

$$\begin{aligned}\dot{V} &= e^T P\dot{e} = e^T P\left(\tilde{f}(x) - c(L+dG)\otimes Ce\right)\\ &\le -ce^T P\left((L+dBG)\otimes C\right)e + \kappa e^T Pe\\ &= -\tfrac{1}{2}ce^T Qe + \kappa e^T Pe\end{aligned}$$

1.4 Discrete-Time Cooperative Tracker

Equation Section (Next)

Now suppose each node or agent has scalar discrete-time (DT) dynamics given by

$$x_i(k+1) = x_i(k) + u_i(k) \tag{1.4.1}$$

with $x_i, u_i \in R$. This corresponds to endowing each node with a memory in the form of a shift register of order one. We discuss two ways to select the control input protocols.

Normalized Protocol DT Systems

Consider the local control protocols

$$u_i(k) = \frac{1}{1+d_i}\sum_{j\in N_i} a_{ij}(x_j(k) - x_i(k)) \tag{1.4.2}$$

which are normalized by dividing by $1+d_i$, with d_i the in-degree of node i. Thus, nodes with larger in-degrees must use less relative control effort than in the protocol **Error! Reference source not found.**. The closed-loop system becomes

$$x_i(k+1) = x_i(k) + \frac{1}{1+d_i}\sum_{j\in N_i} a_{ij}(x_j(k) - x_i(k)) = \frac{1}{1+d_i}\left(x_i(k) + \sum_{j\in N_i} a_{ij}x_j(k)\right) \tag{1.4.3}$$

which is a weighted average of the previous states of node i and its neighbors. The corresponding global input is

$$u(k) = -(I+D)^{-1}Lx(k) \tag{1.4.4}$$

with $D = diag\{d_i\}$. The global dynamics is

$$x(k+1) = x(k) - (I+D)^{-1}Lx(k) = (I+D)^{-1}(I+A)x(k) \equiv Fx(k) \tag{1.4.5}$$

where

$$F = I - (I+D)^{-1}L = (I+D)^{-1}(I+A) \tag{1.4.6}$$

is the normalized DT graph matrix.

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Lee and Spong

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