Lyapunov Techniques in Cooperative Control

Prepared by: F.L. Lewis
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Spong and Chopra

Consider a dynamic graph with node dynamics

$$\dot{x}_i = f_i(x_i) + g_i(x_i)u_i$$

$$y_i = h_i(x_i)$$

which are passive, so that for each *i* there exists a scalar storage function $V(x) \geq 0,\ V(0) = 0$ such that

$$\dot{V} = \left(\frac{\partial V}{\partial x}\right)^T \left(f(x) + g(x)u\right) = L_f V(x) + L_g V(x)u$$

$$= -S(x) + h^T(x)u(x)$$

with $S(x) \geq 0$.

The outputs synchronize if $\lim_{t\to\infty} \left\| y_i(t) - y_j(t) \right\| = 0,\ \forall i, j$.

Linear Protocol

Consider control $u_i = \sum_{j\in N_i} K_{ij}(y_j - y_i)$.

Technical Lemma.

$$\sum_i x_i \sum_j a_{ij}(x_j - x_i) = -\sum_j x_j \sum_i a_{ij}(x_j - x_i)$$

iff the graph is balanced, i.e. iff in-degree= out-degree for all nodes.

Proof: Left-hand side is

$$\sum_i x_i \sum_j a_{ij}(x_j - x_i) = \sum_i x_i \sum_j a_{ij}x_j - \sum_i x_i^2 \sum_j a_{ij}$$

$$= \sum_i \sum_j a_{ij}x_i x_j - \sum_i x_i^2 \sum_j a_{ij}$$

Right-hand side is

$$\sum_j x_j \sum_i a_{ij}(x_j - x_i) = \sum_j x_j^2 \sum_i a_{ij} - \sum_j x_j \sum_i a_{ij}x_i$$

Interchange index variables in first term here to get

$$\sum_j x_j \sum_i a_{ij}(x_j - x_i) = \sum_i x_i^2 \sum_j a_{ji} - \sum_i \sum_j a_{ij}x_i x_j$$

However, the *i*-th row sum $\sum_j a_{ij}$ (in-degree) is equal to the *i*-th column sum $\sum_j a_{ji}$ (out-degree) iff the graph is balanced.

Theorem. Let control graph be connected and balanced (i.e. strongly connected). Then the system is globally stable and the outputs synchronize.

Proof: Consider Lyapunov function candidate

$$V = 2(V_1 + V_2 + \cdots + V_N)$$

With V_i the storage function for state x_i. Then

$$\dot{V} = -2\sum_i S_i(x_i) + 2\sum_i \sum_{j\in N_i} y_i^T K_{ij}(y_j - y_i) = -2\sum_i S_i(x_i) + 2\sum_i \sum_j y_i^T K_{ij}(y_j - y_i)$$

with $S_i(x_i) \geq 0$. According to the technical lemma, the graph is balanced so that

$$\sum_i y_i^T \sum_j a_{ij}(y_j - y_i) = -\sum_j y_j^T \sum_i a_{ij}(y_j - y_i)$$

and

$$\dot{V} = -2\sum_i S_i(x_i) + \sum_i \sum_j y_i^T K_{ij}(y_j - y_i) - \sum_i \sum_j y_j^T K_{ij}(y_j - y_i)$$

$$= -2\sum_i S_i(x_i) - \sum_i \sum_j (y_j - y_i)^T K_{ij}(y_j - y_i) \leq 0$$

LaSalle's invariance principle shows that $\lim_{t\to\infty} \left\| y_i(t) - y_j(t) \right\| = 0$

Connectedness shows that $\lim_{t\to\infty} \left\| y_i(t) - y_j(t) \right\| = 0, \forall i, j$

Nonlinear Protocol

Consider control

$$u_i = \sum_{j\in N_i} \chi(y_j - y_i).$$

where $\chi(z): R \to R$ is a nonlinear function such that:

1. $\chi(z)$ is continuous and locally Lipschitz
2. $\chi(-z) = -\chi(z) \quad \forall z \neq 0$
3. $\chi(y_j - y_i) \equiv \chi_{ji}$ are passive $\forall i, j$, i.e. there exists scalar storage functions $V_{ji}(x) \geq 0,\ V_{ji}(0) = 0,\ S_{ji}(x) \geq 0$ such that

$$\dot{V}_{ji} + S_{ji} = (y_j - y_i)^T \chi(y_j - y_i)$$

Note that for the graph $G = (V, E)$ one can write

$$u_i = \sum_{j\in N_i} \chi(y_j - y_i) = \sum_{(i,j)\in E} \chi(y_j - y_i)$$

Theorem. Let control graph $G = (V, E)$ be undirected and connected. Then the system is globally stable.

Proof: Consider Lyapunov function candidate

$$V = \sum_{i=1}^{N} V_i + \sum_{(i,j)\in E} V_{ji}$$

with V_i the storage function for state x_i. Then

$$\dot{V} = \sum_{i=1}^{N}\left(-S_i(x_i) + y_i^T u_i\right) + \sum_{(i,j)\in E}\left(-S_{ji} + (y_j - y_i)^T \chi(y_j - y_i)\right)$$

$$= -\sum_{i=1}^{N} S_i(x_i) + \sum_{i=1}^{N}\sum_{(i,j)\in E} y_i^T \chi(y_j - y_i) + \sum_{(i,j)\in E}\left(-S_{ji} + (y_j - y_i)^T \chi(y_j - y_i)\right)$$

since

$$\sum_{i=1}^{N}\sum_{j=1}^{N} = \sum_{i=1}^{N}\sum_{j\in N_i} = \sum_{i=1}^{N}\sum_{(i,j)\in E}$$

Since the graph is undirected, for every $y_i^T \chi(y_j - y_i)$ in the second term there is a corresponding $y_j^T \chi(y_i - y_j)$. By property 2 of $\chi(z)$ this is equal to $-y_j^T \chi(y_j - y_i)$. Thus

$$\sum_{i=1}^{N}\sum_{(i,j)\in E} y_i^T \chi(y_j - y_i) = \sum_{(i,j)\in E} y_i^T \chi(y_j - y_i) - \sum_{(i,j)\in E} y_j^T \chi(y_j - y_i) = -\sum_{(i,j)\in E} (y_j - y_i)^T \chi(y_j - y_i)$$

Therefore

$$\dot{V} = -\sum_{i=1}^{N} S_i(x_i) - \sum_{(i,j)\in E} S_{ji} \le 0$$

Therefore the system is globally stable. All trajectories converge to the largest invariant set contained in the set $\dot{V} = 0$, where $\{S_i(x_i) = 0, S_{ji} = 0, \forall (i,j) \in E, \forall i = 1, N\}$

Remark: whether the outputs synchronize depend on the choice of S_i, S_{ji}, and hence on the choice of the control action $\chi(.)$

Sector Nonlinearities

Select $\chi(.)$ to satisfy the additional property

4. $\alpha z^2 \le z\chi(z) \le \beta z^2$ for some $\alpha, \beta > 0$

i.e. sector nonlinearity.

Theorem. Let control graph $G = (V, E)$ be undirected and connected. Then the system is globally stable and the outputs synchronize.

Proof: For the control

$$u_i = \sum_{j\in N_i} \chi(y_j - y_i)$$

one computes

$$S_{ji} = (y_j - y_i)^T \chi(y_j - y_i)$$

Property 2 guarantees that $\{y_i(t) \to y_j(t), \forall (i,j) \in E, \forall i = 1, N\}$. Connectivity guarantees that all outputs synchronize.

Lihua Xie Approach

Define error function as

$$e_i = \sum_j a_{ij}(x_i - x_j) + b_i(x_i - x_0)$$

Then

$$e_i = \sum_j a_{ij}x_i - \sum_j a_{ij}x_j + b_i x_i - b_i x_0$$

$$= d_i x_i - \sum_j a_{ij}x_j + b_i x_i - b_i x_0$$

Define $e = [e_1\ e_2 \cdots e_N]^T$, $x = [x_1\ x_2 \cdots x_N]^T$. Then

$$e = Dx - Ax + \bar{B}x - \bar{B}\underline{1}x_0 = (L + \bar{B})x - \bar{B}\underline{1}x_0$$

with Laplacian $L = D - A$ and $\bar{B} = diag(b_i)$.

Lemma. Let *L* be irreducible and at least one $b_i \neq 0$. Then e=0 iff $x = \underline{1}x_0$.

Proof: Let e=0. Then

$$(L + \bar{B})x = \bar{B}\underline{1}x_0$$

$$(L + \bar{B})x = (L + \bar{B})\underline{1}x_0$$

since $L\underline{1} = 0$. Under the hypotheses, $L + \bar{B}$ is irreducibly diagonally dominant and hence nonsingular. Therefore $x = \underline{1}x_0$

Theorem. Consider first-order node dynamics $\dot{x}_i = u_i$ in an irreducible digraph. Let

$$u_i = (d_i + b_i)^{-1}\left(\sum_j a_{ij}u_j - \alpha e_i\right)$$

for $\alpha > 0$, i.e.

$$u_i = (d_i + b_i)^{-1}\left(\sum_j a_{ij}u_j + \alpha\sum_j a_{ij}(x_j - x_i) + \alpha b_i(x_0 - x_i)\right)$$

Then consensus is reached exponentially at the rate of $e^{-\alpha t}$.

Remark. Note that then

$$u = (D + \bar{B})^{-1}(Au - \alpha e) = (D + \bar{B})^{-1}Au - \alpha(D + \bar{B})^{-1}\left[(L + \bar{B})x - \bar{B}\underline{1}x_0\right]$$

with $u = [u_1\ u_2 \cdots u_N]^T$.

Proof: Note that $I - (D + \bar{B})^{-1}A = (D + \bar{B})^{-1}(L + \bar{B})$, so that

$$(L + \bar{B})u = -\alpha e$$

Select Lyapunov function candidate

$$V = \tfrac{1}{2}e^T e$$

Then

$$\dot{V} = e^T \dot{e} = e^T (L + \bar{B}) \dot{x} = e^T (L + \bar{B}) u$$

$$\dot{V} = -\alpha e^T e = -2\alpha V$$

So that

$$V(t) = e^{-2\alpha t} V(t_0)$$

$$\|e(t)\| = e^{-\alpha t} \|e(t_o)\|$$

For Undirected Graphs

The term $\sum_j a_{ij} u_j$ in $u_i = (d_i + b_i)^{-1} \left(\sum_j a_{ij} u_j - \alpha e_i \right)$ is not nice. We want to get rid of it.

Theorem. Consider first-order node dynamics $\dot{x}_i = u_i$ and let the graph be undirected. Let the graph be irreducible and at least one $b_i \neq 0$. Let

$$u_i = -\alpha (d_i + b_i)^{-1} e_i$$

for $\alpha > 0$, i.e.

$$u_i = \alpha (d_i + b_i)^{-1} \left(\sum_j a_{ij} (x_j - x_i) + b_i (x_0 - x_i) \right)$$

Then consensus is reached exponentially with

$$\|e(t)\| \leq \sqrt{\tfrac{\sigma_{\max}(L+\bar{B})}{\sigma_{\min}(L+\bar{B})}}\, e^{-\beta t} \|e(0)\|$$

where $\beta = \alpha \dfrac{\sigma_{\min}(L + \bar{B})}{\sigma_{\max}(D + \bar{B})}$.

Proof: For undirected graphs Laplacian L is symmetric and positive semidefinite so $L + \bar{B}$ is positive definite. Select Lyapunov function candidate

$$V = \tfrac{1}{2} e^T (L + \bar{B})^{-1} e$$

Then

$$\dot{V} = e^T (L + \bar{B})^{-1} \dot{e} = e^T \dot{x} = e^T u = -a e^T (D + \bar{B})^{-1} e$$

$$\leq -\alpha \sigma_{\min}[(D + \bar{B})^{-1}] \|e\|^2 = -\frac{\alpha}{\sigma_{\max}(D + \bar{B})} \|e\|^2$$

Also

$$V \leq \tfrac{1}{2} \sigma_{\max}[(L + \bar{B})^{-1}] \|e\|^2 = \tfrac{1}{2\sigma_{\min}(L+\bar{B})} \|e\|^2$$

so that

$$\dot{V} \le -\frac{\alpha}{\sigma_{\max}(D+\bar{B})}\|e\|^2 \le -\frac{2\alpha\sigma_{\min}(L+\bar{B})}{\sigma_{\max}(D+\bar{B})}V \equiv -2\beta V$$

Therefore $V(t) \le e^{-2\beta t}V(t_0)$ and

$$\frac{1}{2\sigma_{\max}(L+\bar{B})}\|e\|^2 = \frac{1}{2}\sigma_{\min}[(L+\bar{B})^{-1}]\|e\|^2 \le V$$
$$\le e^{-2\beta t}V(t_0) \le \frac{1}{2\sigma_{\min}(L+\bar{B})}\|e(t_0)\|^2 e^{-2\beta t}$$

So that

$$\|e\|^2 \le \frac{2\sigma_{\max}(L+\bar{B})}{2\sigma_{\min}(L+\bar{B})}e^{-2\beta t}\|e(t_0)\|^2$$
$$\|e(t)\| \le \sqrt{\frac{\sigma_{\max}(L+\bar{B})}{\sigma_{\min}(L+\bar{B})}}\,e^{-\beta t}\|e(t_0)\|$$

Corollary. Under the same setup as in the theorem, Let

$$u_i = -\alpha e_i$$

for $\alpha > 0$, i.e.

$$u_i = \alpha\left(\sum_j a_{ij}(x_j - x_i) + b_i(x_0 - x_i)\right)$$

Then consensus is reached exponentially with

$$\|e(t)\| \le \sqrt{\frac{\sigma_{\max}(L+\bar{B})}{\sigma_{\min}(L+\bar{B})}}\,e^{-\beta t}\|e(0)\|$$

where $\beta = \sigma_{\min}(L+\bar{B})$.

Remark. Convergence is faster without the normalization term.

Using Zhihua Qu Lyapunov Equation

Note that $L+\bar{B}$ is irreducibly diagonally dominant if the graph is irreducible and at least one $b_i \neq 0$, and hence a nonsingular M matrix.

Theorem. Consider first-order node dynamics $\dot{x}_i = u_i$ in an irreducible digraph. Let

$$u_i = -\alpha(d_i + b_i)^{-1}e_i$$

for $\alpha > 0$, i.e.

$$u_i = \alpha(d_i + b_i)^{-1}\left(\sum_j a_{ij}(x_j - x_i) + b_i(x_0 - x_i)\right)$$

Then consensus is reached exponentially with

$$\|e(t)\| \le \sqrt{\frac{\sigma_{\max}(P)}{\sigma_{\min}(P)}} e^{-\beta t}\|e(t_0)\|$$

where

$$\beta = -\frac{\alpha}{2}\frac{\sigma_{\min}(Q)}{\sigma_{\max}(P)}$$

Add in terms of sigma of (L+B^bar)

Remark. Note that then

$$u = -\alpha(D + \bar{B})^{-1}e = -\alpha(D + \bar{B})^{-1}\left[(L + \bar{B})x - \bar{B}\underline{1}x_0\right]$$

with $u = [u_1\ u_2 \cdots u_N]^T$.

Proof: Select Lyapunov function candidate

$$V = \tfrac{1}{2}e^T Pe, \quad P > 0$$

Then

$$\dot{V} = e^T P\dot{e} = e^T P(L + \bar{B})\dot{x} = e^T P(L + \bar{B})u = -\alpha e^T P(L + \bar{B})(D + \bar{B})^{-1}e$$
$$\equiv -\alpha e^T PAe = -\tfrac{1}{2}\alpha e^T\left(PA + A^T P\right)e$$

Note that $A \equiv (L + \bar{B})(D + \bar{B})^{-1}$ is a nonsingular M matrix. Following Zhihua Qu's book, let $w = A^{-1}\underline{1}$ so that $Aw = \underline{1}$. Define $P = diag\{1/w_i\}$. Then

$$A^T P + PA = Q$$

is positive definite and

$$\dot{V} = -\tfrac{1}{2}\alpha e^T Qe$$

so V is a Lyapunov function. Therefore, $e(t) \to 0$ and consensus is reached.

Now

$$\dot{V} \le -\tfrac{1}{2}\alpha\sigma_{\min}(Q)\|e\|^2 \le -\alpha\frac{\sigma_{\min}(Q)}{\sigma_{\max}(P)}V \equiv -2\beta V$$

so that $V(t) \le e^{-2\beta t}V(t_0)$ and

$$\tfrac{1}{2}\sigma_{\min}(P)\left\|e(t)\right\|^2 \le \tfrac{1}{2}\sigma_{\max}(P)e^{-2\beta t}\left\|e(t_0)\right\|^2$$

or

$$\left\|e(t)\right\| \le \sqrt{\frac{\sigma_{\max}(P)}{\sigma_{\min}(P)}}e^{-\beta t}\left\|e(t_0)\right\|$$

Second Order Consensus

Define position and velocity errors

$$e_i = \sum_j a_{ij}(x_i - \Delta_i - x_j + \Delta_j) + b_i(x_i - x_0 - \Delta_i)$$

$$e_{2i} = \sum_j a_{ij}(v_i - v_j) + b_i(v_i - v_0)$$

Then

$$e = (L + \bar{B})(x - \Delta) - \bar{B}\underline{1}x_0$$

$$e_2 = Dv - Av + \bar{B}v - \bar{B}\underline{1}v_0 = (L + \bar{B})v - \bar{B}\underline{1}v_0$$

Lemma. Let L be irreducible and at least one $b_i \neq 0$. Then e=0 iff $x = \underline{1}x_0 + \Delta$, and e_2=0 iff $v = \underline{1}v_0$.

Consider desired consensus dynamics

$$\dot{x}_0 = v_0$$
$$\dot{v}_0 = u_0$$

and second-order node dynamics $\ddot{x}_i = u_i$ or

$$\dot{x}_i = v_i$$
$$\dot{v}_i = u_i$$

Then

$$\dot{e} = e_2$$
$$\dot{e}_2 = (L + \bar{B})u - \bar{B}\underline{1}u_0$$

Define sliding variable $s_i = e_{2i} + \beta e_i$ or $s = e_2 + \beta e$, for $\beta > 0$

Theorem. Asume an irreducible digraph and let

$$u_i = (d_i + b_i)^{-1}\left(\sum_j a_{ij}u_j - \alpha s_i - \beta e_{2i} + \bar{B}\underline{1}u_0\right)$$

for $\alpha > 0$. Then consensus is reached exponentially at the rate of $e^{-\alpha t}$.

Remark. Note that

$$u = (D + \bar{B})^{-1}\left(Au - \alpha s - \beta e_2 + \bar{B}\underline{1}u_0\right)$$

Proof: Note that $I - (D + \bar{B})^{-1}A = (D + \bar{B})^{-1}(L + \bar{B})$, so that

$$(L + \bar{B})u = -\alpha s - \beta e_2 + \bar{B}\underline{1}u_0$$

Select Lyapunov function candidate

$$V = \tfrac{1}{2} s^T s$$

Then

$$\dot{V} = s^T \dot{s} = s^T (\dot{e}_2 + \beta e_2) = s^T ((L + \bar{B})u - \bar{B}\underline{1}u_0 + \beta e_2)$$
$$= -\alpha s^T s = -2\alpha V$$

and the exponential convergence follows.

Ron Chen Pinning Control

Dynamics

$$\dot{x}_i = f(x_i) + u_i$$
$$y_i = \Gamma x_i$$

Control

$$u_i = c\sum_j a_{ij}(y_j - y_i) \; + cd\, b_i(y_0 - y_i) = c\sum_j a_{ij}\Gamma(x_j - x_i) \; + cd\, b_i\Gamma(x_0 - x_i)$$

Define Laplacian $L = D - A = [\ell_{ij}]$ then note that

$$u_i = -c\sum_j \ell_{ij}\Gamma x_j \; + cd\, b_i\Gamma(x_0 - x_i)$$

Then

$$u = -c(D + d\bar{B} - A)\otimes\Gamma x + cd\bar{B}\otimes\Gamma\overline{x}_0 = -c(L + d\bar{B})\otimes\Gamma x + cd\bar{B}\otimes\Gamma\overline{x}_0$$

where $\bar{B} = diag(b_i)$.

$$\dot{x} = f(x) - c(L + d\bar{B})\otimes\Gamma x + cd\bar{B}\otimes\Gamma\overline{x}_0$$

Define error

$$e_i = x_i - x_0$$

Where x_0 is a solution of

$$\dot{x}_0 = f(x_0)$$

Then error dynamics are

$$\dot{e}_i = \dot{x}_i - \dot{x}_0 = f(x_i) - f(x_0) + u_i$$

or

$$\dot{e} = f(x) - f(x_0) - c(L + d\bar{B})\otimes\Gamma x + cd\bar{B}\otimes\Gamma\overline{x}_0$$

Then

$$\dot{e} = f(x) - f(x_0) - c(L + d\bar{B})\otimes\Gamma x + c(d\bar{B} + L)\otimes\Gamma\overline{x}_0$$

$$\dot{e} = f(x) - f(x_0) - c(L + d\bar{B})\otimes\Gamma e$$
$$\equiv \tilde{f}(x) - c(L + d\bar{B})\otimes\Gamma e$$

Lyapunov:

$$V = \tfrac{1}{2}e^T P e$$

$$\dot{V} = e^T P\dot{e} = e^T P\left(\tilde{f}(x) - c(L + d\bar{B}) \otimes \Gamma e\right)$$

$$\begin{aligned}\dot{V} &= e^T P\tilde{f}(x) - ce^T P(L + d\bar{B}) \otimes \Gamma e \\ &= e^T P\tilde{f}(x) - \tfrac{1}{2} ce^T Qe\end{aligned}$$

where

$$P\left(-(L + d\bar{B}) \otimes \Gamma\right) + \left(-(L + d\bar{B}) \otimes \Gamma\right)^T P = -Q$$

since $-(L + d\bar{B}) \otimes \Gamma$ is asymptotically stable if Γ has positive evals.

Note: evals of $A \otimes B$ are $\lambda_i(A)\mu_j(B)$

Case 1: Linearize the dynamics.

Then linearized system is

$$\dot{e} = \left(\nabla f(x_0) - c(L + d\bar{B}) \otimes \Gamma\right)e \equiv Ee$$

and

$$\begin{aligned}\dot{V} &= e^T P\dot{e} = e^T P\left(\nabla f(x) - c(L + d\bar{B}) \otimes \Gamma\right)e \\ &= e^T PEe = \tfrac{1}{2} e^T (PE + E^T P)e = -\tfrac{1}{2} e^T Qe\end{aligned}$$

if E is AS.

Case 2. Dynamics is Lipschitz

Then

$$\left\| f(x) - f(x_0) \right\| \le \kappa \left\| x - x_0 \right\|$$

and

$$\begin{aligned}\dot{V} &= e^T P\dot{e} = e^T P\left(\tilde{f}(x) - c(L + d\bar{B}) \otimes \Gamma e\right) \\ &\le -ce^T P\left((L + d\bar{B}) \otimes \Gamma\right)e + \kappa e^T Pe \\ &= -\tfrac{1}{2} ce^T Qe + \kappa e^T Pe\end{aligned}$$

where

$$P\left(-(L + d\bar{B}) \otimes \Gamma\right) + \left(-(L + d\bar{B}) \otimes \Gamma\right)^T P = -Q$$

Case 3. Dynamics is passive?

Passive node dynamics

$$\dot{x}_i = f_i(x_i) + g_i(x_i)u_i$$

$$y_i = h_i(x_i)$$

Control

$$u_i = c\sum_j a_{ij}(y_j - y_i) \; + cd\, b_i(y_0 - y_i)$$

Now use Spong.

Theorem. Let control digraph be strongly connected. Then the system is globally stable and the outputs synchronize.

Proof: Consider Lyapunov function candidate

$$V = \sum_{i=1}^{N} V_i$$

with V_i the storage function for state x_i. Then

$$\dot{V} = -\sum_i S_i(x_i) + \sum_i y_i^T u_i$$

$$= -\sum_i S_i(x_i) + c\sum_i y_i^T \left(\sum_j a_{ij}(y_j - y_i) \; + cd\, b_i(y_0 - y_i) \right)$$

$$\dot{V} = -2\sum_i S_i(x_i) + y^T(-D + A - \bar{B})y$$

$$= -2\sum_i S_i(x_i) + 2\sum_i \sum_j y_i^T K_{ij}(y_j - y_i)$$

with $S_i(x_i) \geq 0$. According to the technical lemma, the graph is balanced so that

$$\sum_i y_i^T \sum_j a_{ij}(y_j - y_i) = -\sum_j y_j^T \sum_i a_{ij}(y_j - y_i)$$

and

$$\dot{V} = -2\sum_i S_i(x_i) + \sum_i \sum_j y_i^T K_{ij}(y_j - y_i) - \sum_i \sum_j y_j^T K_{ij}(y_j - y_i)$$

$$= -2\sum_i S_i(x_i) - \sum_i \sum_j (y_j - y_i)^T K_{ij}(y_j - y_i) \leq 0$$

LaSalle's invariance principle shows that $\lim_{t\to\infty} \left\| y_i(t) - y_j(t) \right\| = 0$

Connectedness shows that $\lim_{t\to\infty} \left\| y_i(t) - y_j(t) \right\| = 0, \forall i, j$

Lihua Xie Error vs. Ron Chen Disagreement Error

System

$$\dot{x} = f(x) + d + u$$

Lihua local neighborhood error is

$$e_i = \sum_j a_{ij}(x_i - x_j) + b_i(x_i - x_0)$$

$$e = Dx - Ax + Bx - B\underline{1}x_0 = (L+B)x - B\underline{1}x_0$$
$$= (L+B)(x - x_0) = (L+B)\delta$$

where global disagreement error is

$$\delta = x - x_0$$

Note to compute $\delta(t)$ one requires the global state *x* as well as the consensus value x_0. To compute local nbhd error *e(t)* one has all the information available locally.

1. Lihua Error Lyapunov

$$\dot{e} = (L+B)\dot{x} = (L+B)f(x) + (L+B)d + (L+B)u$$

LFC

$$V = \tfrac{1}{2} e^T P e, \quad P > 0$$

Then

$$\dot{V} = e^T P \dot{e} = e^T P(L+B)\dot{x} = e^T P(L+B)(f + d + u)$$
$$= -e^T P(L+B)u + e^T P(L+B)(f + d)$$

$\dot{\delta} = \dot{x} - \dot{x}_0 = \dot{x}$

Select control

$$u = -\alpha e$$

then

$$\dot{V} = -\alpha e^T P(L+B)e + e^T P(L+B)(f + d)$$
$$\equiv -\alpha e^T P(L+B)e + e^T P(L+B)(f + d)$$
$$= -\tfrac{1}{2}\alpha e^T \left(P(L+B) + (L+B)^T P \right) e + e^T P(L+B)(f + d)$$
$$= -\tfrac{1}{2}\alpha e^T Q e + e^T P(L+B)(f + d)$$

2. Disagreement Error

$$\dot{\delta} = \dot{x} - \dot{x}_0 = \dot{x}$$

LFC

$$V = \tfrac{1}{2}\delta^T P\delta$$

Then

$$\dot{V} = \delta^T P\dot{\delta} = \delta^T P\dot{x} = e^T P(f + d + u)$$
$$= -e^T Pu + e^T P(f + d)$$

Select control

$$u = -\alpha e = -\alpha a(L + B)\delta$$

then

$$\dot{V} = -\alpha\delta^T P(L + B)\delta + e^T P(f + d)$$
$$\equiv -\alpha e^T P(L + B)e + e^T P(f + d)$$
$$= -\tfrac{1}{2}\alpha e^T \left(P(L + B) + (L + B)^T P\right)e + e^T P(f + d)$$
$$= -\tfrac{1}{2}\alpha e^T Qe + e^T P(f + d)$$

Note second term does not contain *(L+B)*.