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3. Consensus Under Dynamically Changing Interaction Topologies

We have studied consensus of agents linked together by a communication graph for first-order integrator dynamics and second-order position/velocity dynamics. It has been assumed that the communication graph was constant over time. In practical situations, however, communication links can change with time. An example is the case of formation control, where vehicles are able to communicate only if they are within a certain communication radius of each other. As the vehicles move, the links will change. Therefore, in this section we discuss consensus in teams that have dynamically changing interaction topologies [Jadbabaie et al. 2003, Olfati-Saber and Murray 2004, Ren and Beard 2005, Moreau 2005, Ren and Beard 2008]. The material in this section is from [Jadbabaie et al. 2003, Ren and Beard 2008].

We shall study consensus under time-varying graph topologies for both continuous-time and discrete-time node dynamics. To motivate the upcoming discussion, consider the discrete-time system

$$x(k+1) = F(k)x(k) \tag{3.1}$$

with system matrix $F(k)$ time-varying. The solution of this equation is given by

$$x(k) = F(k-1)\cdots F(1)F(0)x(0) \tag{3.2}$$

which depends on the product of the system matrices. In the time-invariant case $F(k) = F, \forall k$ this becomes simply

$$x(k) = F^k x(0) \tag{3.3}$$

Considering that in Section 1 one had $F = (I + D)^{-1}(I + A)$ with A the adjacency matrix of a graph and D the diagonal matrix of in-degrees, we are motivated by (3.2) to study products of matrices generated by time-varying graph topologies.

Interpret $x(k)$ as a global state $x = [x_1^T \cdots x_N^T]^T$ where $x_i(k)$ are the states of the networked nodes. Then there is a consensus value if and only if

$$\lim_{k\to\infty} F(k-1)\cdots F(1)F(0) = \underline{1}w^T \tag{3.4}$$

for some vector w. For then, (3.2) and (3.4) deliver

$$x_{ss} = \lim_{k\to\infty} F(k-1)\cdots F(1)F(0)x(0) = \underline{1}w^T x(0) \tag{3.5}$$

and all node states go to the consensus value $x_i = x_j = c = w^T x(0), \forall i, j$

Time-Varying Graph Theory

Examine Figure 3.1, which shows three directed communication graphs, none of which is strongly connected or has a spanning tree. Their union, however, is strongly connected. If the

nodes represent dynamical agents, the graphs could represent the communication topology at three subsequent times. Though at no time are all the agents strongly connected, over the complete time period all nodes will eventually communicate with each other.

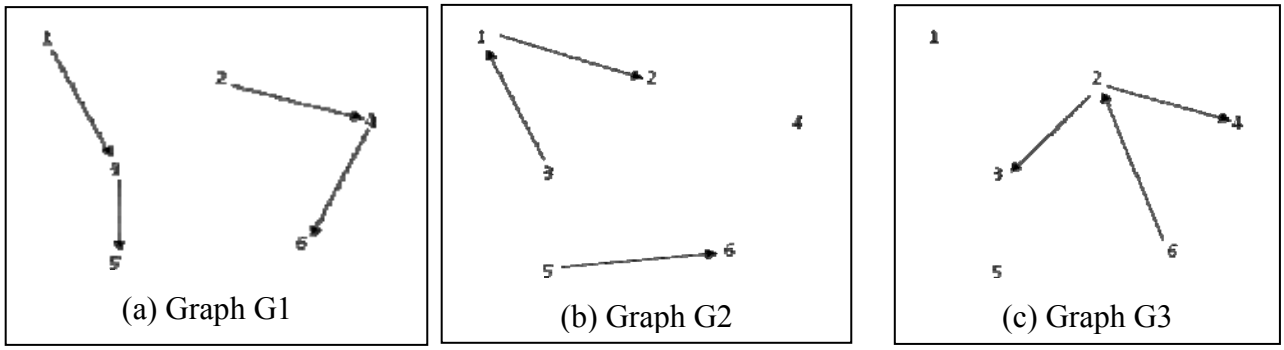


Figure 1. Three disconnected graphs whose union is strongly connected.

Consider a finite set of digraphs $\{G_k, k=1,p\}$. The set is said to be jointly strongly connected if the union of its members is strongly connected. The set is said to jointly have a spanning tree if the union of its members has a spanning tree.

In the study of time-varying graph topologies, the algebras of nonnegative matrices and row stochastic matrices are instrumental. We say a matrix P is nonnegative, $P \geq 0$, if all its elements are nonnegative. Matrix P is positive, $P > 0$, if all its elements are positive. The product of two nonnegative matrices is nonnegative. The product of two nonnegative matrices with positive diagonal elements is nonnegative with positive diagonal elements. Given two nonnegative matrices *P, Q* we say that $P \geq Q$ if $(P-Q) \geq 0$. This is equivalent to saying that each element of *P* is greater than or equal to the corresponding element of *Q*.

Let $G_A = (V_A, E_A), G_B = (V_B, E_B)$ be two graphs. We say G_B is a subgraph of G_A, $G_B \subset G_A$, if $V_B \subset V_A$ and E_B is a subset of the edges E_A restricted to V_B. Given a nonnegative matrix $A = [a_{ij}] \geq 0$, we define the graph G_A corresponding to *A* as the graph having edges $e_{ji}, i \neq j$ when $a_{ij} > 0, i \neq j$, and corresponding edge weights a_{ij}. Note that the diagonal elements of *A* play no role in defining G_A since we assume graphs are simple, that is, have no loops and no multiple edges between the same points. Let *A* be the adjacency matrix of a graph G_A and *B* be the adjacency matrix of a graph G_B, with *A, B* indexed by the same vertex set. Then $A \geq \alpha B$ for some scalar $\alpha > 0$ implies that $G_B \subset G_A$, because the matrix inequality implies that for every nonzero entry b_{ij} of *B* there is a corresponding nonzero entry a_{ij} of *A*. That is, for every edge in G_B there is an edge in G_A.

Lemma 1. Let $p > 2$ and $P_1, \cdots P_p$ be nonnegative matrices with positive diagonal elements. Then

$$P_1 P_2 \cdots P_p \geq \gamma (P_1 + P_2 + \cdots + P_p) \tag{3.6}$$

for some $\gamma > 0$

Proof: [Jadbabaie et al. 2003]. Let μ, ρ be respectively the minimum and maximum diagonal

elements of all the matrices P_i. Define $\delta = \mu^2 / 2\rho$. Then one can write each matrix as $P_i = \mu I + B_i$ where $B_i \succeq 0$. Then for any two matrices one has

$$P_i P_j = (\mu I + B_i)(\mu I + B_j) = \mu^2 I + \mu(B_i + B_j) + B_i B_j \succeq \mu^2 I + \frac{\mu^2}{2\rho}(B_i + B_j)$$
$$= \delta((\rho I + B_i) + (\rho I + B_j)) \succeq \delta(P_i + P_j)$$

The proof therefore follows by induction. It is found that $\gamma = (\mu^2 / 2\rho)^{p-1}$. ■

This lemma shows that the graph corresponding to the sum of the matrices $\{P_i\}$ is a subgraph of the graph corresponding to their product.

Let matrix $F \in R^{N \times N}$ be row stochastic, that is all row sums are equal to one. The product of two row stochastic matrices *F, G* is matrices is row stochastic because $FG\underline{1} = F\underline{1} = \underline{1}$. Matrix *F* is said to be stochastically irreducible and aperiodic (SIA), or ergodic, if

$$\lim_{k \to \infty} F^k = \underline{1} w^T \tag{3.7}$$

for some $w \in R^N$, with $\underline{1} \in R^N$ the vector of 1's. The next results show when a row stochastic matrix is SIA.

Lemma 2. [Horn and Johnson 1985]. Let $F = [f_{ij}] \in R^{N \times N}$ be row stochastic, $F\underline{1} = \underline{1}$, and *w* satisfy $w^T F = w^T$, with $w^T \underline{1} = 1$. If all other eigenvalues of *F* have magnitudes less than 1, then (3.7) holds. ■

This result shows that, under the hypothesis, *w* in (3.7) is a normalized left eigenvector of *F* for $\lambda_1 = 1$. In the next result, matrix *F* is associated with a graph.

Lemma 3. [Ren and Beard 2008]. Given a graph $G = (V, E)$ with *N* nodes, let $F = [f_{ij}] \in R^{N \times N}$ be nonnegative and row stochastic, and satisfy $f_{ij} > 0 \Leftrightarrow e_{ji} \in E$. Then $F\underline{1} = \underline{1}$, that is $\lambda_1 = 1$ is an eigenvalue of *F* with right eigenvector equal to $\underline{1}$. Moreover, $\lambda_1 = 1$ is simple, that is, it has multiplicity equal to 1, if and only if graph G has a spanning tree. Furthermore, if $f_{ii} > 0, \forall i$, all other eigenvalues of *F* have magnitude less than 1. ■

These results show that if *F* is a row stochastic matrix with diagonal elements $f_{ii} > 0, \forall i$, and its graph has a spanning tree, then *F* is SIA. Clearly, if the graph is strongly connected, it has a spanning tree and *F* is SIA.

The next result from [Wolfowitz 1963] is important in the study of changing graph topologies.

Lemm 4. [Wolfowitz]. Let $\{S_1, \cdots, S_p\}$ be a finite set of SIA matrices with the property that, for each sequence $S_{i_1}, S_{i_2}, \cdots, S_{i_j}$ of positive length, the product $S_{i_j} \cdots S_{i_2} S_{i_1}$ is SIA. Then for every infinite sequence $S_{i_1}, S_{i_2}, \cdots$ there exists a vector *w* such that

$$\lim_{j\to\infty} S_{i_j} \cdots S_{i_2} S_{i_1} = \underline{1} w^T \tag{3.8}$$

■

In this lemma, the requirement that the set of SIA matrices be finite is crucial for the proof.

Consensus for Discrete-Time Systems Under Varying Graph Topology

Consider discrete-time node dynamics given by

$$x_i(k+1) = x_i(k) + u_i(k) \tag{3.9}$$

with $x_i, u_i \in R$. Select the normalized local voting control protocols

$$u_i(k) = \frac{1}{1+d_i} \sum_{j\in N_i} a_{ij}(x_j(k) - x_i(k)) \tag{3.10}$$

and take the communication over a graph with adjacency matrix *A*. This yields the global dynamics

$$x(k+1) = x(k) - (I+D)^{-1} L x(k) = (I+D)^{-1}(I+A)x(k) \equiv Fx(k) \tag{3.11}$$

with $u = [u_1 \cdots u_N]^T \in R^N$, $x = [x_1 \cdots x_N]^T \in R^N$, where

$$F = I - (I+D)^{-1} L = (I+D)^{-1}(I+A) \tag{3.12}$$

is the normalized DT graph matrix and *D* is the diagonal matrix of in-degrees. Clearly, F satisfies the hypotheses of Lemma 3.

The following analysis is for the case of scalar states and controls $x_i, u_i \in R$, but it generalizes easily to the vector case $x_i, u_i \in R^n$ by using the Kronecker product.

We wish to allow the graph interactions to vary with time, so let

$$x(k+1) = F(k)x(k) \tag{3.13}$$

with corresponding time-varying adjacency matrices $A(k)$, in-degree matrices $D(k)$, and graphs $G(k)$. Note that the graph topology need not switch at every time index *k*. Given a finite set of *N* nodes, the set $\bar{G}$ of possible graph topologies is finite.

Theorem 1. **Discrete-Time Consensus on Changing Graph Topologies.** [Jadbabaie et al. 2003, Ren and Beard 2008]. Consider the dynamics on switched graphs (3.13). Assume there exists an infinite sequence of contiguous, nonempty, bounded time intervals $[k_j, k_{j+1})$, $j = 0,1,\cdots$ starting at $k_0 = 0$, such that the union of graphs over each interval has a spanning tree. Then consensus is reached and

$$x_{ss} = \underline{1} c \tag{3.14}$$

for some consensus value *c*.

Proof:

The set of all possible DT graph matrices $F(k)$ is finite because the graph is finite and has a finite number of nodes and edges. Let the DT graph matrices on interval $[k_j, k_{j+1})$ be

$\{F(k): k \in \{k_j, k_j+1, \cdots, k_{j+1}-1\}\}$. Each $F(k) = [f_{ij}] \in R^{N\times N}$ is nonnegative and row stochastic, has $f_{ii} > 0$, and satisfies $f_{ij} > 0 \Leftrightarrow e_{ji} \in E(k)$ for its corresponding graph $G(k)$. From Lemma 1 one has over the interval $[k_j, k_{j+1})$

$$F[k_j] \equiv F(k_{j+1}-1)\cdots F(k_j+1)F(k_j) \geq \gamma(F(k_j) + F(k_j+1) + \cdots + F(k_{j+1}-1))$$

for some $\gamma > 0$. The union of graphs over each interval has a spanning tree, so the sum on the right-hand side has a spanning tree. Therefore, the product $F[k_j]$ has a spanning tree. The product of row stochastic matrices is row stochastic. The product of nonnegative matrices with positive diagonal elements is nonnegative with positive diagonal elements is. Therefore, from Lemma 3, $F[k_j]$ has $\lambda_1 = 1$ is simple with all other eigenvalues having magnitude less than 1. Lemma 2 shows that $F[k_j]$ is SIA.

Now the Wolfowitz Lemma 4 shows that

$$\lim_{j\to\infty} F[k_j]\cdots F[k_1]F[k_0] = \underline{1}w^T$$

for some $w \in R^N$. The solution of (3.13) is (3.2). Define $k_{j+1} < k$ as the largest time for which the union of graphs over $[k_j, k_{j+1})$ has a spanning tree. Then

$$x(k) = F(k-1)\cdots F(k_{j+1}+1)F(k_{j+1})F[k_j]\cdots F[k_1]F[k_0]x(0)$$

So that

$$\lim_{k\to\infty} x(k) = F(k-1)\cdots F(k_{j+1}+1)F(k_{j+1})F[k_j]\cdots F[k_1]F[k_0]x(0)$$
$$= F(k-1)\cdots F(k_{j+1}+1)F(k_{j+1})\underline{1}w^T x(0)$$

Since all matrices are row stochastic one has

$$\lim_{k\to\infty} x(k) = \underline{1}w^T x(0)$$

and consensus is reached. ■

Note that in the limit $x_i = x_j = c, \forall i, j$ with the consensus value given by

$$c = w^T x(0)$$

The vector *w* depends on the set of graphs switched between as well as the switching times. It cannot be predicted ahead a priori. It should be related in some fashion to the left eigenvectors for $\lambda_1 = 1$ of the set of graphs.

This result has been extended to case where the union of graphs over bounded, non-overlapping, non-contiguous intervals has a spanning tree in [Ren and Beard 2005].

Consensus for Continuous-Time Systems Under Varying Graph Topology

Consider scalar single-integrator node dynamics

$$\dot{x}_i = u_i \tag{3.15}$$

with $x_i, u_i \in R$. Consider the local control protocols

$$u_i = \sum_{j \in N_i} a_{ij}(x_j - x_i) \tag{3.16}$$

on a graph with adjacency matrix $A = [a_{ij}]$. This yields the global dynamics

$$\dot{x} = -Dx + Ax = -(D - A)x = -Lx \tag{3.17}$$

with $u = [u_1 \cdots u_N]^T \in R^N$, $x = [x_1 \cdots x_N]^T \in R^N$, where D is the diagonal matrix of in-degrees.

We wish to allow the graph interactions to vary with time. Consider a finite set of possible graph topologies $\{G_p : p = 1, \cdots, P\}$ with G_p having adjacency matrix A_p, in-degree matrix D_p, and Laplacian L_p. Let

$$\dot{x} = -L_\sigma x \tag{3.18}$$

with $\sigma(t) \in \{1, 2, \cdots, P\}$ a switching signal whose value at time t indicates which graph describes the current topology at time t. Let $t_0, t_1, \cdots$ be the times at which $\sigma(t)$ switches and let $t_{i+1} - t_i > \tau$ for some fixed dwell time $\tau > 0$.

The solution to (3.18) on the interval $[t_i, t_{i+1})$ is given by

$$x(t) = e^{-L_\sigma t} x(t_i), t \in [t_i, t_{i+1}) \tag{3.19}$$

To show consensus for system (3.18) on time-varying graph topologies, we must study the relationship between the Laplacian matrix L and the transition matrix e^{-Lt}.

Lemma 4. [Ren and Beard 2008]. Let $L = D - A \in R^{N \times N}$ be the Laplacian matrix of a directed graph G_A. Then e^{-Lt} is a nonnegative row stochastic matrix with positive diagonal elements. Moreover, $\mu_1 = 1$ is a simple eigenvalue for e^{-Lt} and all other eigenvalues of e^{-Lt} are strictly inside the unit circle if and only if G_A has a spanning tree.

Proof: Note that $e^{-Lt} = (I - Lt - L^2 t^2 / 2! - \cdots)$ so that $e^{-Lt} \underline{1} = \underline{1}$ and e^{-Lt} is row stochastic. Define $M = \beta I - L$ with β the maximum in-degree, i.e. the maximum element of D. Then M is nonnegative. Write $e^{-Lt} = e^{-\beta t} e^{Mt} = e^{-\beta t}(I + Mt + M^2 t^2 / 2! + \cdots)$ and note that the product of two nonnegative matrices is nonnegative. Therefore, e^{-Lt} is a nonnegative matrix with positive diagonal elements.

Let λ_i be the eigenvalues of L. If the graph has a spanning tree, then $\lambda_1 = 0$ is a simple eigenvector for L with all other eigenvalues having positive real parts. The eigenvalues of e^{-Lt} are $\mu_i = e^{-\lambda_i t}$. The map $z = e^{st}$ maps the left half s-plane into the unit circle in the z-plane. Consequently, if G_A has a spanning tree, $\mu_1 = 1$ is a simple and all other eigenvalues of e^{-Lt} are inside the unit circle. ■

Lemma 5. [Ren and Beard 2008]. Let $L = D - A \in R^{N \times N}$ be the Laplacian matrix of a directed graph G_A. Then the graph $G_L = G_A$ corresponding to L is a subgraph of the graph G_ϕ corresponding to $\phi(t) = e^{-Lt}$.

Proof: Define $M = \beta I - L$ with β the maximum in-degree. Then M is nonnegative. Write $e^{-Lt} = e^{-\beta t}e^{Mt} = e^{-\beta t}(I + Mt + M^2t^2/2! + \cdots)$ so that $e^{-Lt} \succeq \alpha M$ for some $\alpha > 0$. Therefore, $G_M \subset G_\phi$. However, $G_L = G_M$. ■

The next result shows when the continuous time local voting protocol delivers consensus on switched graph topologies. In connection with this proof, the importance of the setup in (3.18) is that the Laplacians are constant between switchings, and $t_j - t_{j-1} > \tau$ for some fixed dwell time $\tau > 0$ so that the number of graphs in any bounded time interval is finite.

Theorem 2. **Continuous-Time Consensus on Changing Graph Topologies.** Consider the dynamics on switched graphs (3.18). Let the graphs switch at times $t_0, t_1, \cdots$. Assume there exists an infinite sequence of contiguous, nonempty, bounded time intervals $[t_{i_j}, t_{i_{j+1}})$, $j = 1, 2, \cdots$ starting at $t_{i_1} = t_0 = 0$, such that the union of graphs over each interval has a spanning tree. Then consensus is reached and

$$x_{ss} = \underline{1}c \tag{3.20}$$

for some consensus value *c*.

Proof: Let $\tau_i = t_{i+1} - t_i$ and denote the Laplacian matrix on interval $[t_i, t_{i+1})$ by $L(t_i)$. Define the corresponding state transition matrices $\phi(t_i) = e^{-L(t_i)\tau_i}$. On the interval $[t_{i_j}, t_{i_{j+1}})$ one has the set of Laplacians $\{L(t_{i_j}), L(t_{i_j+1}), \cdots, L(t_{i_{j+1}-1})\}$ whose union has a spanning tree. According to Lemma 5, the union of $\{\phi(t_{i_j}), \phi(t_{i_j+1}), \cdots, \phi(t_{i_{j+1}-1})\}$ has a spanning tree. From Lemma 1 one has over the interval $[t_{i_j}, t_{i_{j+1}})$

$$\phi[t_{i_j}] \equiv \phi(t_{i_{j+1}-1}) \cdots \phi(t_{i_j+1})\phi(t_{i_j}) \succeq \gamma(\phi(t_{i_j}) + \phi(t_{i_j+1}) + \cdots + \phi(t_{i_{j+1}-1}))$$

for some $\gamma > 0$.

Write the solution of (3.18) as

$$x(t) = e^{-L(t-t_{i_j})}\phi[t_{i_j}] \cdots \phi[t_{i_1}]\phi[t_{i_0}]x(0)$$

where t_{i+1} is the largest time for which the union of graphs over $[t_i, t_{i+1})$ has a spanning tree. Now identify $\phi[t_{i_j}]$ with $F[k_j]$ in the proof of Theorem 1 and follow the steps there to see that consensus is reached. ■

This result has been extended to case where the graph weights are time-varying between switchings in [Ren and Beard 2008], and to the case where the union of graphs over bounded, non-overlapping, non-contiguous intervals has a spanning tree in [Ren and Beard 2005].

Gossip Algorithms and Other Extensions

We have assumed that all nodes update their states according to the local voting protocols (3.10)

or (3.16) at each time instant. There are several variants of this. In Markov decision processes, only a few nodes, generally selected at random, are updated at each time. In gossip algorithms, at each time one selects a random node for update. That node randomly selects one of its neighbors and updates its own state based on the difference between its state and that of the selected neighbor.

The next results show results for convergence to consensus.

Corollary 1. **Random Node Updates.** Consider either the discrete-time consensus protocol (3.9), (3.10) or the continuous-time protocol with dwell times as in (3.16), (3.18). Let the graph have a spanning tree. At each time select a node at random and perform the update for that node based on all of its neighbors. Then, consensus is reached if each node is selected for update an infinite number of times. ■

For the following, we define an interaction pair as a pair of nodes (v_i, v_j) such that e_{ji} is an edge of the graph.

Corollary 2. **Gossip Algorithms.** Consider either the discrete-time consensus protocol (3.9), (3.10) or the continuous-time protocol with dwell times as in (3.16), (3.18). Let the graph have a spanning tree. At each time select a node v_i at random. Select one of that node's neighbors v_j at random. Perform the update of the state of node v_i based on only the interaction pair (v_i, v_j). Then, consensus is reached if each interaction pair corresponding to each edge of the graph is selected for update an infinite number of times. ■

The proofs of these results rely on the fact that, under the stated conditions, there exists an infinite sequence of contiguous, nonempty, bounded time intervals such that the union of graphs over each interval has a spanning tree.

Gossip Algorithm for Average Consensus

Consider time-invariant graphs with a spanning tree. It has been seen in previous sections that, under the discrete-time protocol (3.9), (3.10) one converges to the consensus value

$$c = \sum_{i=1}^{N} p_i x_i(0) \tag{3.21}$$

where $w_1 = \begin{bmatrix} p_1 & \cdots & p_N \end{bmatrix}^T$ is the normalized left eigenvector of the DT graph matrix F for $\lambda_1 = 1$. Under the continuous-time protocol (3.15), (3.16) one converges to the consensus value (3.21) where $w_1 = \begin{bmatrix} p_1 & \cdots & p_N \end{bmatrix}^T$ is the normalized left eigenvector of the graph Laplacian L for $\lambda_1 = 0$. If the graph is balanced, one reaches average consensus

$$c = \tfrac{1}{N} \sum_{i=1}^{N} x_i(0) \tag{3.22}$$

In some applications it is required to converge to average consensus regardless of graph topology. It is not desired for the consensus value to depend on how the nodes communicate to reach it. An average consensus algorithm can easily be developed using gossip methods.

Consider a communication graph containing a spanning tree and discrete-time node dynamics (3.9). At each update time, select an interaction pair as of nodes (v_i, v_j) corresponding to an edge e_{ji}. Update the states of both nodes based on the state difference according to

$$x_i = x_i + \frac{x_j - x_i}{2}, \qquad x_j = x_j - \frac{x_j - x_i}{2} \tag{3.23}$$

Note that updating of both nodes simultaneously implies an undirected graph structure.

Corollary 3. **Gossip Algorithm for Average Consensus.** Let the graph contain a spanning tree and perform gossip algorithm (3.23) at each time for a randomly selected pair of nodes (v_i, v_j) corresponding to an edge e_{ji} of the graph. Then, the nodes reach average consensus (3.22) regardless of the graph topology as long as the interaction pair (v_i, v_j) corresponding to every edge e_{ji} of the graph is selected for update an infinite number of times.

Proof: Under the stated condition, there exists an infinite sequence of contiguous, nonempty, bounded time intervals such that the union of graphs over each interval has a spanning tree. Therefore, Theorem 1 provides the result. Now note that, using (3.23), the sum $(x_i + x_j)$ is invariant. ■

4. Consensus With Communication Delays

So far we have assumed that each node can obtain information from all its neighbor states with no time delay. In practical applications, however, the graph topology might represent a communication network, with the edges representing the allowed communication links between agents. These links might be subject to two problems- links may fail or new links may be added, and there may be communication delays. In the previous section we considered the case of changing graph topologies. Here, we discuss the case of communication delays along the graph links.

Protocol With Constant Communication Link Delays

Take the scalar single-integrator node dynamics

$$\dot{x}_i = u_i \tag{4.1}$$

with $x_i, u_i \in R,\ i = 1, N$. Consider the local control protocols

$$\dot{x}_i = u_i(t) = \sum_{j \in N_i} a_{ij}(x_j(t - \tau_{ij}) - x_i(t)) \tag{4.2}$$

on a graph with adjacency matrix $A = [a_{ij}]$. This includes the communication delay τ_{ij} along the link a_{ij} in receiving information from neighbor node *j*. This corresponds to edges with weights $h_{ij}(s) = a_{ij}e^{-s\tau_{ij}}$, so that the edges are now dynamical systems with transfer functions $h_{ij}(s)$.

To write the global dynamics in a convenient form, assume that all delays are constant and identical so that $\tau_{ij} = \tau > 0, \forall i, j$. Then the global dynamics are

$$\dot{x}(t) = -Dx(t) + Ax(t - \tau) \tag{4.3}$$

with $u = [u_1 \cdots u_N]^T \in R^N$, $x = [x_1 \cdots x_N]^T \in R^N$, where *D* is the diagonal matrix of in-degrees.

This system is stable for any positive time delay [Moreau 2004]. At steady-state one has

$$0 = -Dx_{ss} + Ax_{ss} = -Lx_{ss} \tag{4.4}$$

So that x_{ss} is in the nullspace of Laplacian matrix L. If the graph has a directed spanning tree, then *L* has rank *N-1*, $\lambda_1 = 0$ is simple, and $N(L) = c\underline{1}$ for some constant $c > 0$. Then, the states reach consensus, $x_i(t) \to x_j(t), \forall i, j$.

Let *w* be a first left eigenvector of *L*, so that $w^T L = 0$. Then

$$w^T \dot{x}(t) = -w^T Dx(t) + w^T Ax(t - \tau) \neq 0 \tag{4.5}$$

so that the quantity $w^T x(t)$ is not conserved along the system trajectories. Consequently, the consensus value reached is not generally a weighted average of the initial states of the nodes.

Average-Conserving Protocol for Link Delays

Now consider the local control protocols

$$\dot{x}_i = u_i(t) = \sum_{j \in N_i} a_{ij}(x_j(t-\tau_{ij}) - x_i(t-\tau_{ij})) \tag{4.6}$$

where the delay is also applied to the node's own state. Take constant identical delays along all links so that the global dynamics are

$$\dot{x}(t) = -Lx(t-\tau) \tag{4.7}$$

Let $w = \begin{bmatrix} p_1 & p_2 & \cdots & p_N \end{bmatrix}^T$ be a first left eigenvector of L so that $w^T L = 0$. Then

$$w^T \dot{x}(t) = -w^T L x(t-\tau) = 0 \tag{4.8}$$

so that the quantity $w^T x(t)$ is conserved along the system trajectories. Consequently, the consensus value reached is the weighted average of initial states

$$x_{ss} = \frac{\sum_i p_i x_i(0)}{\sum_i p_i} \tag{4.9}$$

If the graph is balanced, then $w = \underline{1}$ and the consensus value is the average of the initial states

$$x_{ss} = c = \frac{\sum_i p_i x_i(0)}{\sum_i p_i} \tag{4.10}$$

It is shown in [Olfati-Saber and Murray 2003] that for undirected connected graphs, (4.8) is asymptotically stable and consensus is reached if

$$\tau < \frac{\pi}{2\lambda_N} \tag{4.11}$$

with λ_N the maximum eigenvalue of Laplacian matrix L.

According to the discussion in Section 1 based on the Gerschgorin Circle Criterion, all eigenvalues of L are inside a circle centered at $d_{\max}$ with radius $d_{\max}$, where $d_{\max}$ is the maximum in-degree of the graph. Therefore, $\lambda_N < 2d_{\max}$ and a sufficient condition for consensus is

$$\tau < \frac{\pi}{4d_{\max}} \tag{4.12}$$

This means that graphs with large nodes cannot tolerate large communication delays.

Performance Tradeoffs and Robustness

Recall from Section 1 that for large values of t one has

$$x(t) \to v_2 e^{-\lambda_2 t} w_2^T x(0) + \underline{1}\, c \tag{4.13}$$

with c the consensus value in (4.10), λ_2 the Fiedler eigenvalue, and w_2 the second left eigenvector of L. Thus, it is interesting that the time constant to reach consensus using is given by $1/\lambda_2$, whereas the maximum delay tolerated is given by $\pi / 2\lambda_N$. A bound on the Fiedler eigenvalue is given by

$$\lambda_2 \leq \frac{N}{N-1} d_{\min} \tag{4.14}$$

with $d_{\min}$ the minimum in-degree. Thus, for fast convergence one would like to see a large $d_{\min}$, but for good tolerance to delays one would like to see a small $d_{\max}$.

A communication network can be made more robust to delays. Consider the modified protocol

$$\dot{x}_i = u_i(t) = k \sum_{j \in N_i} a_{ij} (x_j(t - \tau_{ij}) - x_i(t - \tau_{ij})) \tag{4.15}$$

with $k > 0$ a control gain. Then,

$$\dot{x}(t) = -kLx(t - \tau) \tag{4.16}$$

corresponds to a graph with Laplacian kL, which has a largest eigenvalue of $k\lambda_N$. Now, condition (4.11) becomes

$$\tau < \frac{\pi}{2k\lambda_N} \tag{4.17}$$

so that by decreasing the control gains *k*, arbitrarily large communication delays can be tolerated. This means that the state derivatives $\dot{x}_i$ change more slowly and it takes longer to reach consensus. Note that the Fiedler eigenvalue of (4.16) is equal to $k\lambda_2$ so that the time constant to reach consensus is now $1/k\lambda_2$.

The ratio of maximum allowed delay to time constant is given by $\pi\lambda_2 / 2\lambda_N$. We would like this ratio to be large. The ration λ_2 / λ_N has been called the eigenratio. It is a measure of functionality of a network divided by interconnection costs [Varshney 2010].

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