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Bipartite consensus of multi-agent systems over signed graphs: State feedback and output feedback control approaches

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SUMMARY

This paper studies bipartite consensus problems for continuous-time multi-agent system over signed directed graphs. We consider general linear agents and design both state feedback and dynamic output feedback control laws for the agents to achieve bipartite consensus. Via establishing an equivalence between bipartite consensus problems and the conventional consensus problems under both state feedback and output feedback control approaches, we make direct application of existing state feedback and output feedback consensus algorithms to solve bipartite consensus problems. Moreover, we propose a systematical approach to design bipartite consensus control laws. Copyright © 2016 John Wiley & Sons, Ltd.

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KEY WORDS: bipartite consensus; multi-agent system; signed graph

1. INTRODUCTION

In the past few years, there has been tremendous interest in developing distributed control laws for multi-agent systems with a primary focus on consensus over nonnegative graphs [1–5]. Nonnegative graphs are defined by edges with nonnegative weights, which are appropriate for describing collaborative relations between agents. Broad applications along this line of research include formation of aircrafts or unmanned ground vehicles and wireless sensor networks. When both collaborative and antagonistic interactions coexist within a group of agents, nonnegative graphs cease to be applicable. Instead, the underlying communication networks can be more suitably represented by signed graphs, in which a positive edge means collaboration and a negative edge represents an antagonistic interaction. Research on collective behaviors over signed graphs has been recently addressed in the literature [6–11], and finds applications in scenarios of social networks [12], predator–prey dynamics [13], biological systems [14], and so on.

Altafini studied bipartite consensus problem over signed graphs in [6], where it was found that two subgroups of single-integrator agents form during evolution, and consensus is achieved within each subgroup, moving individually towards opposite directions. Bipartite consensus of single-integrator agents was also studied in [8], where both homogeneous signed graphs and heterogeneous signed graphs are considered. However, single-integrator dynamics cannot describe a system with multiple states, for example, position and velocity. Bipartite flock of multiple double-integrator agents was considered in [7]. More recently, the work [6] was further extended to state feedback control of linear time-invariant (LTI) single-input systems [9], and the communication graph is assumed to be undirected, connected, and structurally balanced, wherein it was shown that stabilizability

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of the system matrix pair (A, b) is necessary and sufficient for the agents to achieve bipartite consensus under a state feedback control law. As a natural extension in the sense of both node dynamics and graph topology, in this paper we study bipartite consensus problem for a general multi-input multi-output (MIMO) LTI systems over directed signed graphs from a new perspective. We observe that for MIMO LTI systems, interestingly, bipartite consensus over signed graphs is equivalent to conventional consensus over certain nonnegative graphs, under both state feedback and output feedback control. In light of this equivalence, a systematic approach is then proposed to derive control laws for bipartite consensus problems from established control laws for conventional consensus problems. A class of control protocols are then designed to solve the bipartite consensus problem, which are modified from the work [15], where consensus tracking problems over nonnegative digraphs are considered. We note that the reference [16] also studies the bipartite consensus problem of MIMO LTI systems, in which a different output feedback control law is proposed. However, the signed graph is required to be weight balanced. In our current work, this condition is removed. Preliminary results of this paper have been presented at conferences [10, 11, 17].

This paper is organized as follows. Preliminaries on graph theory are presented in Section 2. Bipartite consensus problem is formulated and its relation with conventional consensus problem is established in Section 3. A class of state feedback and output feedback control laws for bipartite consensus problem are proposed in Section 4 with illustrating simulation examples in Section 5. Section 6 concludes the paper.

2. PRELIMINARIES ON SIGNED GRAPH

2.1. Notations

Notations in this paper are rather standard. The empty set is $\varnothing$. A matrix with entries a_{ij} is denoted by $[a_{ij}]$, while $[A]_{ij}$ is the entry of the i-th row and the j-th column of matrix A. A diagonal matrix with entries $\sigma_1, \ldots, \sigma_n$ is $diag(\sigma_1, \ldots, \sigma_n)$. The identity matrix is $I_N \in \mathbb{R}^{N\times N}$. A vector with all ones is $\mathbf{1}_N = [1, \ldots, 1]^T \in \mathbb{R}^N$. The Kronecker product and the Cartesian product are denoted by $\otimes$ and $\times$, respectively. The signum function is $sgn(\cdot)$ with the definition

$$sgn(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

2.2. Signed graph

Communication network of a multi-agent system can be modeled by a graph, represented as $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, with the node set $\mathcal{V} = \{v_1, \ldots, v_N\}$ and the edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. We use v_i to denote node i. An edge (v_j, v_i), graphically depicted by an arrow tailed at node j and headed at node i, means that node i can receive information of node j, and node j is called a neighbor of node i. For the edge (v_j, v_i), we also call node j the parent, and i the child. Let N_i denote the neighbor set of node i, that is, $N_i = \{j \mid (v_j, v_i) \in \mathcal{E}\}$. Let a_{ij} be the weight associated with edge (v_j, v_i). By a positive/negative edge (v_j, v_i), we mean that its associated weight a_{ij} is positive/negative. If there is no edge from node j to node i, one has $a_{ij} = 0$. Then the topology of a graph can be fully captured by its adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$. We use $\mathcal{G}(\mathcal{A})$ to explicitly denote a graph whose adjacency matrix is $\mathcal{A}$. A graph is undirected if $\mathcal{A} = \mathcal{A}^T$, and directed (known as digraph) if otherwise. Obviously, a undirected graph is a special case of a directed graph. A nonnegative graph means that all its edges are positive, while a signed graph means that each edge can be either positive or negative. Clearly, nonnegative graph is a special case of signed graph. A path from node i to node j is a sequence of edges $\{(v_i, v_l), (v_l, v_p), \ldots, (v_q, v_j)\}$. A cycle is a path that begins and ends at the same node. For a digraph, a semi-cycle is a sequence of edges that turn into a undirected cycle when neglecting directions of all the edges. A cycle or a semi-cycle is positive (negative) if the product of all its edge weights is positive (negative). A digraph is a directed spanning tree if each node has only one parent, except for one node, called the root, which does not have any parents.

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A digraph has a directed spanning tree if deleting some edges properly leaves a directed spanning tree. A digraph is strongly connected if there is a directed path between each ordered pair of nodes. An undirected graph is connected if there is a path between each pair of nodes. In this paper, we consider directed signed graphs and assume that $a_{ii} = 0$.

Two Laplacian matrices are used in this paper. One is the *conventional* Laplacian matrix

$$L_c = diag\left(\sum_{j=1}^{N} a_{1j}, \ldots, \sum_{j=1}^{N} a_{Nj}\right) - \mathcal{A}, \tag{1}$$

which plays a vital role in studying consensus problem over nonnegative graphs [2, 5]. The other is adopted from [6] and is defined as

$$L_s = diag\left(\sum_{j=1}^{N} |a_{1j}|, \ldots, \sum_{j=1}^{N} |a_{Nj}|\right) - \mathcal{A}, \tag{2}$$

which is important when analyzing the collective behavior over signed graphs. An interesting graph topology specific to signed graphs is called structural balance.

Definition 1 (Structural balance [6, 18]*)*
A signed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ is structurally balanced if it has a bipartition of two nonempty node sets $\mathcal{V}_p$ and $\mathcal{V}_q$, with the property that $\mathcal{V}_p \cup \mathcal{V}_q = \mathcal{V}$ and $\mathcal{V}_p \cap \mathcal{V}_q = \varnothing$, such that $a_{ij} \geqslant 0$ when v_i and v_j are in the same subgroup, and $a_{ij} \leqslant 0$ otherwise.

A structurally balanced signed graph is always associated with a nonnegative graph through the so-called signature matrix, which is shown in the following result. It is adopted from [6, Lemma 2] by observing that the requirement of strong connectedness is not needed.

Lemma 1
Denote the signature matrices set as

$$\mathcal{D} = \{diag(\sigma_1, \ldots, \sigma_N) \mid \sigma_i \in \{1, -1\}\}.$$

Then a signed directed graph $\mathcal{G}(\mathcal{A})$ is structurally balanced if and only if any of the following conditions holds:

(a) the corresponding undirected graph $\mathcal{G}(\mathcal{A}_u)$ is structurally balanced, where $\mathcal{A}_u = \mathcal{A} + \mathcal{A}^T$, and $a_{ij}a_{ji} \geqslant 0$;
(b) $\exists D \in \mathcal{D}$, such that $\bar{\mathcal{A}} = [\bar{a}_{ij}] = D\mathcal{A}D$ is a nonnegative matrix; and
(c) either there are no semi-cycles or all semi-cycles are positive.

Proof

(a) Structural balance implies that $a_{ij}a_{ji} \geqslant 0$. Thus, $sgn(a_{ij}) = sgn\left([\mathcal{A}_u]_{ij}\right)$, and statement (a) follows. Similar development justifies the sufficiency.
(b) By Definition 1, we construct D by picking $\sigma_i = 1$ for all i such that $v_i \in \mathcal{V}_p$, and $\sigma_j = -1$ for those $v_j \in \mathcal{V}_q$. When nodes i and j are in the same subgroup, that is, $a_{ij} \geqslant 0$ and $\sigma_i\sigma_j = 1$, we have $\bar{a}_{ij} = \sigma_i\sigma_j a_{ij} \geqslant 0$. Similarly, we also have $\bar{a}_{ij} \geqslant 0$ when nodes i and j are in different subgroups. Therefore, $\bar{\mathcal{A}}$ is always a nonnegative matrix. Conversely, $\bar{a}_{ij} = \sigma_i\sigma_j a_{ij} \geqslant 0$ implies that when nodes i and j are in the same subgroup, that is, $\sigma_i\sigma_j = 1$, one has $a_{ij} \geqslant 0$; when they are in different subgroups, one has $a_{ij} \leqslant 0$. This meets the definition of structural balance.
(c) When there are no semi-cycles, the graph $\mathcal{G}(\mathcal{A})$ either is a directed spanning tree or has isolated subgroup(s). By noticing [6, Corollary 1] and Definition 1, one can easily conclude that such digraph is structurally balanced. When all semi-cycles of $\mathcal{G}(\mathcal{A})$ are positive, all cycles of $\mathcal{G}(\mathcal{A}_u)$ are positive. By [6, Lemma 1], all isolated subgroup(s) of $\mathcal{G}(\mathcal{A}_u)$ are structurally balanced.

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Then it is easy to observe that $\mathcal{G}(\mathcal{A}_u)$ and thus $\mathcal{G}(\mathcal{A})$ is structurally balanced. Suppose that there exists a negative semi-cycle of $\mathcal{G}(\mathcal{A})$. Then $\mathcal{G}(\mathcal{A}_u)$ also has a negative cycle. By [6, Lemma 1], graph $\mathcal{G}(\mathcal{A}_u)$ and hence $\mathcal{G}(\mathcal{A})$ is structurally unbalanced. □

It is well known that when a nonnegative digraph has a directed spanning tree, 0 is a simple eigenvalue of its conventional Laplacian matrix L_c and all its other eigenvalues have positive real parts [5]. A parallel result for signed digraphs is given as follows.

Lemma 2 ([8, 10]*)*
If a signed directed graph $\mathcal{G}(\mathcal{A})$ has a directed spanning tree and is structurally balanced, then 0 is a simple eigenvalue of L_s and all its other eigenvalues have positive real parts, but not vice versa.

Proof
When $\mathcal{G}(\mathcal{A})$ is structurally balanced, according to Lemma 1, $\mathcal{G}(\bar{\mathcal{A}})$ is nonnegative and has a directed spanning tree. This implies that 0 is a simple eigenvalue of the conventional Laplacian matrix $\bar{L}_c$ of the graph $\mathcal{G}(\bar{\mathcal{A}})$ and all its other eigenvalues have positive real parts. Moreover, it is trivial to show that L_s and $\bar{L}_c$ have the same spectrum. The converse need not to be true. A counterexample can be easily conceived. □

Note that part of this result is shown in [6, Lemma 2], where the graph is required to be strongly connected. For this subtle difference, our proof also differs.

Corollary 1 ([10]*)*
Suppose that the undirected signed graph $\mathcal{G}(\mathcal{A})$ is connected. Then it is structurally balanced, if and only if 0 is a simple eigenvalue of L_s and all its other eigenvalues are positive.

Corollary 2 ([10]*)*
Suppose that the nonnegative digraph $\mathcal{G}(\mathcal{A})$ has a directed spanning tree. Then for any $D \in \mathcal{D}$, the graph $\mathcal{G}(D\mathcal{A}D)$ is a signed digraph, has a spanning tree, and is structurally balanced.

3. EQUIVALENCE BETWEEN BIPARTITE CONSENSUS AND CONVENTIONAL CONSENSUS

In this section, we formulate the bipartite consensus problem and show an equivalence between bipartite consensus and conventional consensus under certain state feedback control and output feedback control laws. Moreover, we propose a systematic way to construct bipartite consensus control laws from the well-studied conventional consensus problems.

3.1. Problem formulation

Consider a group of agents, each modeled by a LTI system

$$\dot{x}_i = Ax_i + Bu_i, \quad y_i = Cx_i, \qquad i = 1, \ldots, N \tag{3}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, and $y_i \in \mathbb{R}^q$ are the state, input, and output, respectively; the triple (A, B, C) is controllable and observable. The communication network is depicted by a signed digraph $\mathcal{G}(\mathcal{A})$, which is assumed to have a directed spanning tree and be structurally balanced.

Definition 2 (Bipartite consensus)
System (3) is said to achieve bipartite consensus if there exists some nontrivial trajectory $x^*(t)$, such that $\lim_{t\to\infty} x_i(t) = x^*(t)$, $\forall i \in p$ and $\lim_{t\to\infty} x_j(t) = -x^*(t)$, $\forall j \in q$, where $p \cup q = \{1, \ldots, N\}$ and $p \cap q = \varnothing$.

Evidently, when either set p or q is empty, bipartite consensus reduces to the well-known conventional *consensus* [5], where all nodes converge to the same value, that is, $\lim_{t\to\infty} \big(x_i(t) - x_j(t)\big) = 0$, $\forall i, j$.

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3.2. Equivalence under state feedback control

To facilitate the presentation, we first define bipartite consensus problem and conventional consensus problem under state feedback control.

Problem 1 (State feedback bipartite consensus)
Consider system (3) over a signed digraph $\mathcal{G}(\mathcal{A})$, which has a directed spanning tree and is structurally balanced. Design a distributed state feedback control law

$$u_i = K \sum_{j \in N_i} a_{ij}(x_j - sgn(a_{ij})x_i) \tag{4}$$

where K is the control gain matrix, such that bipartite consensus is achieved.

Problem 2 (State feedback consensus)
Consider the following system

$$\dot{z}_i = Az_i + Bv_i, \qquad \omega_i = Cz_i, \qquad i = 1, \ldots, N. \tag{5}$$

over a nonnegative digraph $\mathcal{G}(\bar{\mathcal{A}})$, which has a directed spanning tree. Design a distributed state feedback control law

$$v_i = K \sum_{j \in N_i} \bar{a}_{ij}(z_j - z_i) \tag{6}$$

where K is the control gain matrix, such that conventional consensus is achieved.

The equivalence between these two problems is proved in the following theorem.

Theorem 1
For LTI multi-agent system, state feedback bipartite consensus (e.g., Problem 1) and state feedback consensus (e.g., Problem 2) are equivalent, if the nonnegative graph $\mathcal{G}(\bar{\mathcal{A}})$ is associated with the signed graph $\mathcal{G}(\mathcal{A})$ in the sense that $\bar{\mathcal{A}} = [\bar{a}_{ij}] = D\mathcal{A}D$, where D is chosen in accordance with Lemma 1. In other words, the state feedback control gain K that solves Problem 1 can also solve Problem 2, and vice versa.

Proof
For Problem 1, the closed-loop system can be collectively written as

$$\dot{x} = (I_N \otimes A - L_s \otimes BK)x, \tag{7}$$

where $x = [x_1^T, \ldots, x_N^T]^T$ and L_s is the Laplacian matrix of $\mathcal{G}(\mathcal{A})$. For Problem 2, the closed-loop systems is

$$\dot{z} = (I_N \otimes A - \bar{L}_c \otimes BK)z, \tag{8}$$

where $z = [z_1^T, \ldots, z_N^T]^T$ and $\bar{L}_c$ is the conventional Laplacian matrix of $\mathcal{G}(\bar{\mathcal{A}})$. Because $\bar{\mathcal{A}} = D\mathcal{A}D$, we have $\bar{L}_c = DL_sD$.

We shall abuse the notation z and define a state transformation $z = (D \otimes I_n)x$. Following (7), we have

$$\dot{z} = (D \otimes I_n)\dot{x} = (I_N \otimes A - \bar{L}_c \otimes BK)z.$$

This is exactly the closed-loop form (8) of Problem 2. Because $z = (D \otimes I_n)x$, we have $x_i(t) = \sigma_i z_i(t)$, $\sigma_i \in \{1, -1\}$; that is, conventional consensus of z_i is equivalent to bipartite consensus of x_i. Therefore, the control gain K, which solves Problem 1, can also solve Problem 2. The converse part can be similarly obtained following Corollary 2. □

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3.3. Equivalence under output feedback control

Now, we establish the equivalence of bipartite consensus and conventional consensus problems of LTI systems under a certain output feedback control law. Define

$$\tilde{y}_i = y_i - \hat{y}_i,$$

$$\tilde{\omega}_i = \omega_i - \hat{\omega}_i.$$

Problem 3 (Output feedback bipartite consensus)
Consider system (3) over a signed digraph $\mathcal{G}(\mathcal{A})$, which has a directed spanning tree and is structurally balanced. Design an output feedback control law

$$u_i = K \sum_{j \in N_i} a_{ij} \left(\hat{x}_j - sgn(a_{ij}) \hat{x}_i \right), \tag{9}$$

$$\dot{\hat{x}}_i = A\hat{x}_i + Bu_i - F\tilde{y}_i, \tag{10}$$

where K and F are control gain matrices, such that bipartite consensus is achieved.

Problem 4 (Output feedback consensus)
Consider system (5) over a nonnegative digraph $\mathcal{G}(\bar{\mathcal{A}})$, which has a directed spanning tree. Design an output feedback control law

$$v_i = K \sum_{j \in N_i} \bar{a}_{ij} (\hat{z}_j - \hat{z}_i), \tag{11}$$

$$\dot{\hat{z}}_i = A\hat{z}_i + Bv_i - F\tilde{\omega}_i, \tag{12}$$

with K and F being the control gain matrices, such that conventional consensus is achieved.

Theorem 2
For LTI multi-agent system, output feedback bipartite consensus (e.g., Problem 3) and output feedback consensus (e.g., Problem 4) are equivalent, if the nonnegative graph $\mathcal{G}(\bar{\mathcal{A}})$ is associated with the signed graph $\mathcal{G}(\mathcal{A})$ in the sense that $\bar{\mathcal{A}} = [\bar{a}_{ij}] = D\mathcal{A}D$, where D is chosen in accordance with Lemma 1.

Proof
The proof follows the similar spirit of that for Theorem 1, thus is omitted for brevity. □

Remark 1
Theorem 2 presents the equivalence between bipartite consensus problem and conventional consensus problem under a certain output feedback control. Similarly, equivalence should also hold under other output feedback control laws, such as those modified from [15, Sections V.A and V.C].

3.4. A systematic approach to solve bipartite consensus control problem

Consider bipartite consensus problem of system (3) over a signed digraph $\mathcal{G}(\mathcal{A})$, which has a directed spanning tree and is structurally balanced. Inspired by the equivalence, a systematic approach to solve bipartite consensus problem is proposed as follows.

Step 1 Define σ_i as in Lemma 1, and define transformations $x_i = \sigma_i z_i$, $y_i = \sigma_i \omega_i$, $u_i = \sigma_i v_i$, $a_{ij} = \sigma_i \sigma_j \bar{a}_{ij}$. System (3) over a signed graph $\mathcal{G}(\mathcal{A})$ can be transformed into the following multi-agent system

$$\dot{z}_i = Az_i + Bv_i, \quad \omega_i = Cz_i, \qquad i = 1, \ldots, N \tag{13}$$

over the corresponding nonnegative graph $\mathcal{G}(\bar{\mathcal{A}})$, where $\bar{\mathcal{A}} = [\bar{a}_{ij}] = D\mathcal{A}D$. Note that system (3) and system (13) have identical dynamics (A, B, C).

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Step 2 Design a state feedback control law

$$v_i = f_1(z_i, z_{j|_{j\in N_i}}, \bar{a}_{ij}) \tag{14}$$

or a dynamic output feedback control law

$$v_i = f_2(\omega_i, \omega_{j|_{j\in N_i}}, \hat{z}_i, \hat{z}_{j|_{j\in N_i}}, \bar{a}_{ij}) \tag{15}$$

$$\dot{\hat{z}}_i = f_3(\omega_i, \omega_{j|_{j\in N_i}}, \hat{z}_i, \hat{z}_{j|_{j\in N_i}}, \bar{a}_{ij}) \tag{16}$$

for system (13) to achieve conventional consensus, where $\hat{z}_i$ is the estimate of z_i, and $\hat{z}_j$ the estimate of z_j.

Step 3 Substituting $z_i = \sigma_i x_i$, $z_j = \sigma_j x_j$, $\omega_i = \sigma_i y_i$, $\omega_j = \sigma_j y_j$, $\hat{z}_i = \sigma_i \hat{x}_i$, $\hat{z}_j = \sigma_j \hat{x}_j$, $v_i = \sigma_i u_i$, $\bar{a}_{ij} = \sigma_i \sigma_j a_{ij}$ back into (14) or (15)–(16) yields a state feedback control law

$$u_i = g_1(x_i, x_{j|_{j\in\mathcal{N}_i}}, a_{ij})$$

or an output feedback control law

$$u_i = g_2(y_i, y_{j|_{j\in N_i}}, \hat{x}_i, \hat{x}_{j|_{j\in N_i}}, a_{ij}) \tag{17}$$

$$\dot{\hat{x}}_i = g_3(y_i, y_{j|_{j\in N_i}}, \hat{x}_i, \hat{x}_{j|_{j\in N_i}}, a_{ij}), \tag{18}$$

which will solve the bipartite consensus problem of system (3) over the structurally balanced signed graph $\mathcal{G}(\mathcal{A})$.

4. CONTROLLER DESIGN FOR BIPARTITE CONSENSUS PROBLEMS

By observing the equivalence presented in Section 3, a state feedback consensus tracking protocol proposed in [15, Section III] is applied to solve bipartite consensus problem of system (3) with slight modification.

Corollary 3
Consider system (3) over signed digraph $\mathcal{G}(\mathcal{A})$, which has a directed spanning tree and is structurally balanced. Let

$$u_i = cK \sum_{j\in N_i} a_{ij}(x_j - sgn(a_{ij})x_i), \tag{19}$$

where

- $c > 0$ is a scalar control gain satisfying

$$c \geqslant \frac{1}{2\min_{i\in\mathcal{I}} Re(\lambda_i)} \tag{20}$$

  with $Re(\lambda_i)$ being the real part of the i-th eigenvalue of the Laplacian matrix L_s of $\mathcal{G}(\mathcal{A})$, and $\mathcal{I} = \{i \mid Re(\lambda_i) > 0,\ i \in \{1, \ldots, N\}\}$;
- K is a matrix control gain and takes the form

$$K = R^{-1}B^T P \tag{21}$$

  with P being the unique positive definite solution of the algebraic Riccati equation

$$A^T P + PA + Q - PBR^{-1}B^T P = 0,$$

  where Q and R are both positive-definite design matrices with appropriate dimensions.

Then system (3) achieves bipartite consensus.

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Proof
First, we shown that controller

$$v_i = cK \sum_{j \in N_i} \bar{a}_{ij}(z_j - z_i), \tag{22}$$

with the same c and K as in (20) and (21), achieves conventional consensus of system (5) over nonnegative digraph $\mathcal{G}(\bar{\mathcal{A}})$, where $\bar{\mathcal{A}} = [\bar{a}_{ij}] = D\mathcal{A}D$, and D is chosen in accordance with Lemma 1. The closed-loop system is

$$\dot{z} = (I_N \otimes A - c\bar{L}_c \otimes BK)z. \tag{23}$$

Because graph $\mathcal{G}(\bar{\mathcal{A}})$ is nonnegative and has a directed spanning tree, the eigenvalues $\bar{\lambda}_i$ of $\bar{L}_c$ satisfy $0 = \bar{\lambda}_1 < Re(\bar{\lambda}_2) \leqslant \cdots \leqslant Re(\bar{\lambda}_N)$. To simplify the notations, we shall replace $\bar{\lambda}_i$ with λ_i without changing any results, because $\bar{L}_c$ and L_s are similar matrices. There exists a nonsingular matrix $M = [m_1, \ldots, m_N] \in \mathbb{R}^{N \times N}$, where $m_1 = \mathbf{1}_N$ is a right eigenvector of $\bar{L}_c$ associated with the eigenvalue 0, such that $J = M^{-1}\bar{L}_c M = diag(0, J_{N-1})$ is a Jordan form of $\bar{L}_c$. Note that $J_{N-1} \in \mathbb{R}^{(N-1)\times(N-1)}$ is itself a Jordan form with nonzero diagonal entries $\lambda_2, \cdots, \lambda_N$.

Define $q = [q_1^T, q_2^T, \ldots, q_N^T]^T = [q_1^T,]^T = (M^{-1} \otimes I_n)z$. Then following (23), we have

$$\dot{q}_1 = Aq_1,$$

$$\dot{\mathfrak{q}} = (I_{N-1} \otimes A - cJ_{N-1} \otimes BK)\mathfrak{q} = \bar{A}_c\mathfrak{q}.$$

Matrix $\bar{A}_c$ is a block diagonal or block upper-triangular matrix with diagonal entries $A - c\lambda_i BK$ $(i = 2, \ldots, N)$. Following the same technique as used in [15, Theorem 1], we can show that $A - c\lambda_i BK$ are Hurwitz for all $i = 2, \ldots, N$, and thus, $\bar{A}_c$ is Hurwitz. Therefore, we have $\lim_{t\to\infty}(t) = 0$, that is, $\lim_{t\to\infty} q_i(t) = 0$, $i = 2, \ldots, N$. Because $z(t) = (M \otimes I_n)q = \sum_{k=1}^{N}(m_k \otimes I_n)q_k$, finally we have $\lim_{t\to\infty} z(t) = \lim_{t\to\infty}(m_1 \otimes I_n)q_1(t)$, that is,

$$\lim_{t\to\infty} z_i(t) = \lim_{t\to\infty} e^{At}q_1(0), \quad \forall i = 1, \ldots, N.$$

Then bipartite consensus of system (3) follows from the equivalence property in Theorem 1. □

A class of dynamic output feedback cooperative tracking control laws in [15, Section V] can be modified to solve conventional consensus problems, and thus bipartite consensus problems of system (3), because of the equivalence presented in Section 3. This is shown as follows.

Corollary 4
Consider system (3) over graph $\mathcal{G}(\mathcal{A})$. Let

$$u_i = cK \sum_{j \in N_i} a_{ij}\left(\hat{x}_j - sgn(a_{ij})\hat{x}_i\right), \tag{24}$$

$$\dot{\hat{x}}_i = A\hat{x}_i + Bu_i - cF\tilde{y}_i, \tag{25}$$

where

- c and K are chosen as in (20) and (21), respectively, and
- F is designed such that $(A + cFC)$ is Hurwitz.

Then system (3) achieves bipartite consensus.

Corollary 5
Consider system (3) over graph $\mathcal{G}(\mathcal{A})$. Let

$$u_i = cK \sum_{j \in N_i} a_{ij}(\hat{x}_j - sgn(a_{ij})\hat{x}_i), \tag{26}$$

$$\dot{\hat{x}}_i = A\hat{x}_i + Bu_i - cF \sum_{j \in N_i} a_{ij}(\tilde{y}_j - sgn(a_{ij})\tilde{y}_i), \tag{27}$$

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where

- c and K are chosen as in (20) and (21), respectively, and
- F is designed as

$$F = P_1 C^T R_1^{-1} \tag{28}$$

with P_1 being the unique positive-definite solution of the following algebraic Riccati equation

$$AP_1 + P_1 A^T + Q_1 - P_1 C^T R_1^{-1} C P_1 = 0,$$

where Q_1 and R_1 are both positive-definite design matrices with appropriate dimensions.

Then system (3) achieves bipartite consensus.

Corollary 6
Consider system (3) over graph $\mathcal{G}(\mathcal{A})$. Let

$$u_i = K\hat{x}_i, \tag{29}$$

$$\dot{\hat{x}}_i = A\hat{x}_i + Bu_i - cF \sum_{j \in N_i} a_{ij}(\tilde{y}_j - sgn(a_{ij})\tilde{y}_i) \tag{30}$$

where

- c and F are chosen as in (20) and (28), respectively, and
- K is designed such that $A + BK$ is Hurwitz.

Then system (3) achieves bipartite consensus.

The proofs for Corollaries 4–6 take advantage of the equivalence property and follow similar procedure as in Corollary 3, thus is omitted for brevity. It should be noted that many state feedback and output feedback control laws for consensus problem can be extended in a similar way to solve the corresponding bipartite consensus problems, such as the one in [3]. Section 4 can be regarded as applications of the equivalence property.

5. NUMERICAL EXAMPLES

Consider a multi-agent system with six nodes, and each node is modeled by a general LTI system with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -3 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

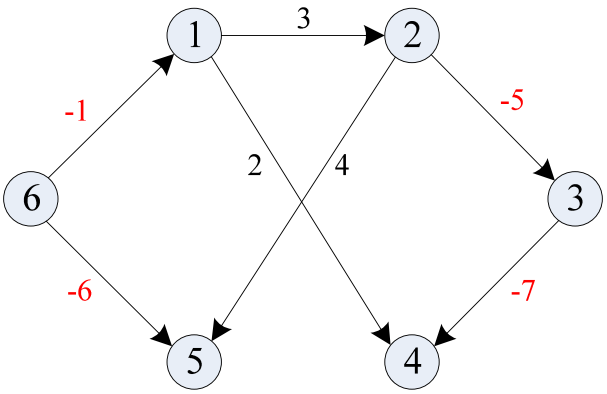


Figure 1. Signed digraph.

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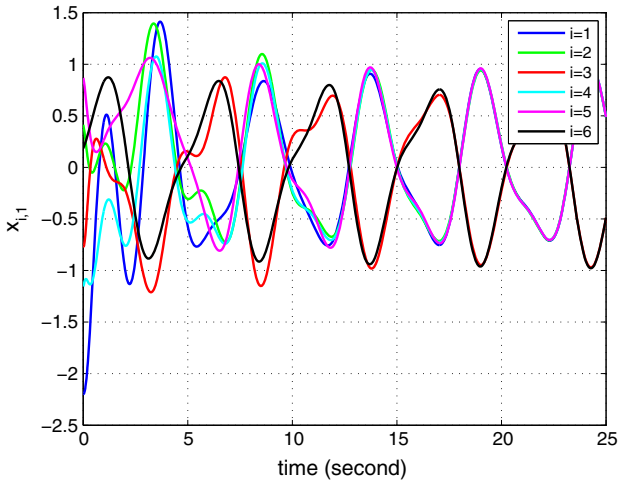


Figure 2. Example for Corollary 3.

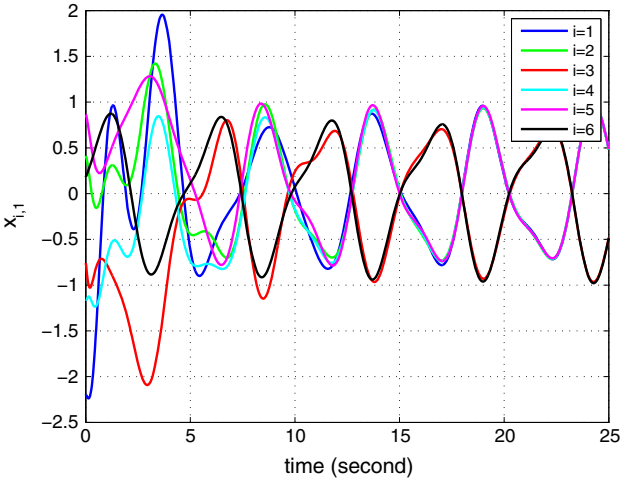


Figure 3. Example for Corollary 4.

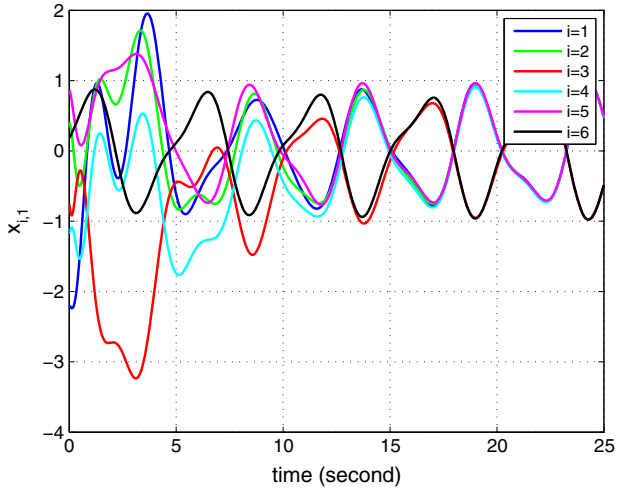


Figure 4. Example for Corollary 5.

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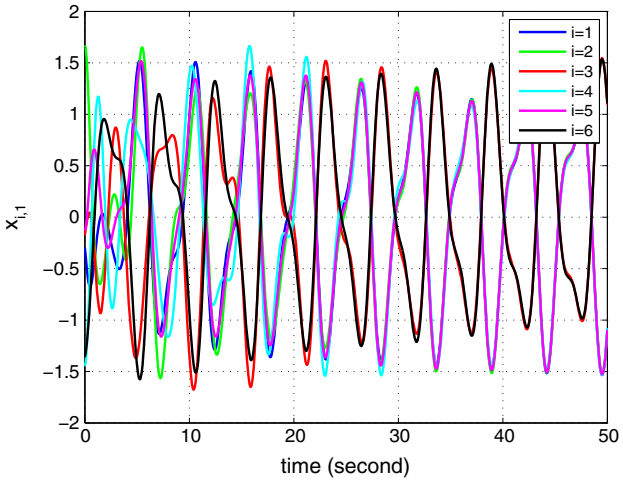


Figure 5. Example for Corollary 6.

When the signed digraph has a directed spanning tree and is structurally balanced (Figure 1), bipartite consensus is achieved under all four control laws proposed in Section 4, as shown in Figures 2–5, respectively. The group separates into two subgroups, i.e., v_1, v_2, v_4, v_5 and v_3, v_6. We only put the state trajectories of $x_{i,1}$ for illustration.

6. CONCLUSION

In this paper, we investigated bipartite consensus of general LTI multi-agent systems over directed signed graphs, where both collaborative and antagonistic interactions coexist. We solve this problem from a new perspective, that is, by establishing an equivalence between bipartite consensus and conventional consensus, under both state feedback and output feedback control. Thus, many state/output feedback control laws for consensus problems can be applied to solve bipartite consensus problem over signed graphs. It would be interesting to further investigate this equivalence in a more general setup, such as heterogeneous nonlinear systems [19, 20].

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