2.5 Cartesian and Other Dynamics

In Section 2.2 we derived the robot dynamics in terms of the time behavior of $q(t)$. According to Table 2.3-1,

$$M(q)\ddot{q} + V(q,\dot{q}) + F_v\dot{q} + F_d(\dot{q}) + G(q) + \tau_d = \tau \tag{2.5-1}$$

or

$$M(q)\ddot{q} + N(q,\dot{q}) + \tau_d = \tau, \tag{2.5-2}$$

where the nonlinear terms are

$$N(q,\dot{q}) \equiv V(q,\dot{q}) + F_v\dot{q} + F_d(\dot{q}) + G(q). \tag{2.5-3}$$

We call this the dynamics of the arm formulated in joint space, or simply the *joint-space dynamics.*

Cartesian Arm Dynamics

It is often useful to have a description of the dynamical development of variables other than the joint variable $q(t)$. Consequently, define

$$y = h(q) \tag{2.5-4}$$

with $h(q)$ a generally nonlinear transformation. Although $y(t)$ could be any variable of interest, let us think of it here as the Cartesian or task space position of the end effector (i.e., position and orientation of the end effector in base coordinates).

The derivation of the Cartesian dynamics from the joint-space dynamics is akin to the feedback linearization in Section 2.4. Differentiating (2.5-4) twice yields

$$\dot{y} = J\dot{q} \tag{2.5-5}$$

$$\ddot{y} = J\ddot{q} + \dot{J}\dot{q}, \tag{2.5-6}$$

where the Jacobian is

$$J \equiv \frac{\partial h}{\partial q}. \tag{2.5-7}$$

The Cartesian velocity vector is $\dot{y} = [v^T \quad \omega^T]^T \in \mathbb{R}^6$, with $v \in \mathbb{R}^3$ the linear velocity and $\omega \in \mathbb{R}^3$ the angular velocity. Let us assume that the number of links is $n = 6$, so that J is square. Assuming also that we are away from workspace singularities so that $|J| \neq 0$, according to (2.5-6), we may write

$$\ddot{q} = J^{-1}\ddot{y} - J^{-1}\dot{J}\dot{q}, \tag{2.5-8}$$

which is the "inverse acceleration" transformation. Substituting this into (2.5-2) yields

$$MJ^{-1}\ddot{y} + (N - MJ^{-1}\dot{J}\dot{q}) + \tau_d = \tau.$$

Recalling now the force transformation $\tau = J^T F$, with F the Cartesian force vector (see Appendix A) we have

$$J^{-T}MJ^{-1}\ddot{y} + J^{-T}(N - MJ^{-1}\dot{J}\dot{q}) + J^{-T}\tau_d = F. \tag{2.5-9}$$

This may be written as

$$\overline{M}\ddot{y} + \overline{N} + f_d = F, \tag{2.5-10}$$

where we have defined the Cartesian inertia matrix, nonlinear terms, and disturbance by

$$\overline{M} \equiv J^{-T}MJ^{-1} \tag{2.5-11}$$

$$\overline{N} \equiv J^{-T}(N - MJ^{-1}\dot{J}\dot{q}) = J^{-T}(N - MJ^{-1}\dot{J}J^{-1}\dot{y}) \tag{2.5-12}$$

$$f_d \equiv J^{-T}\tau_d. \tag{2.5-13}$$

Equation (2.5-9)–(2.5-10) gives the Cartesian or workspace dynamics of the robot manipulator.

Note that $\overline{M}$, $\overline{N}$, and f_d depend on q and $\dot{q}$, so that strictly speaking, the Cartesian dynamics are not completely given in terms of $y, \dot{y}, \ddot{y}$. However, $\dot{q} = J^{-1}\dot{y}$, and given $y(t)$ we could use the inverse kinematics to determine $q(t)$, so that $\overline{M}$, $\overline{N}$, f_d can be computed as functions of y and $\dot{y}$ using computer subroutines.

Structure and Properties of the Cartesian Dynamics

It is important to realize that all the properties of the joint-space dynamics listed in Table 2.3-1 carry over to the Cartesian dynamics as long as J is nonsingular [Slotine 1987]. Note particularly that $\overline{M}$ is symmetric and positive definite. For a revolute arm the Jacobian has units of length and is bounded. In that case, $\overline{M}$ is bounded above and below.

Defining

$$\overline{V} \equiv J^{-T}(V(q,\dot{q}) - MJ^{-1}\dot{J}J^{-1}\dot{y}), \tag{2.5-14}$$

it follows that

$$\overline{V} = \overline{V}_m \dot{y} \tag{2.5-15}$$

with

$$\overline{V}_m = J^{-T}(V_m - MJ^{-1}\dot{J})J^{-1}, \tag{2.5-16}$$

where V_m was defined in Section 2.3.

It is easy to show that

$$\overline{S} \equiv \dot{\overline{M}} - 2\overline{V}_m \tag{2.5-17}$$

is skew-symmetric. Indeed, use the identity

$$\frac{d}{dt}(J^{-1}) = -J^{-1}\dot{J}J^{-1} \tag{2.5-18}$$

to see that

$$\begin{aligned}\dot{\overline{M}} - 2\overline{V}_m &= \frac{d}{dt}(J^{-T})MJ^{-1} + J^{-T}\dot{M}J^{-1} + J^{-T}M\frac{d}{dt}(J^{-1}) - 2\overline{V}_m \\ &= -J^{-T}\dot{J}^{T}J^{-T}MJ^{-1} - J^{-T}MJ^{-1}\dot{J}J^{-1} + 2J^{-T}MJ^{-1}\dot{J}J^{-1} \\ &\quad + J^{-T}(\dot{M} - 2V_m)J^{-1} \\ &= J^{-T}(\dot{M} - 2V_m + [MJ^{-1}\dot{J} - (MJ^{-1}\dot{J})^{T}])J^{-1},\end{aligned}$$

which is skew symmetric since $\dot{M} - 2V_m$ is.

The friction terms in the Cartesian dynamics are

$$\overline{F}_v\dot{y} + \overline{F}_s \equiv J^{-T}F_vJ^{-1}\dot{y} + J^{-T}F_d(\dot{q}), \tag{2.5-19}$$

and they satisfy bounds like those in Table 2.3-1. Notice that in Cartesian coordinates the friction effects are not decoupled (e.g., $J^{-T}F_vJ^{-1}$ is not diagonal). The Cartesian gravity vector

$$\overline{G} \equiv J^{-T}G(q) \tag{2.5-20}$$

is bounded.

The property of linearity in the parameters holds and is expressed as

$$\overline{M}\ddot{y} + \overline{N} = \overline{M}\ddot{y} + \overline{V}_m\dot{y} + \overline{F}_v\dot{y} + \overline{F}_s + \overline{G} = \overline{W}(y,\dot{y},\ddot{y})\varphi, \tag{2.5-21}$$

where the known Cartesian function of robot functions is

$$\overline{W}(y,\dot{y},\ddot{y}) = J^{-T}W(q,\dot{q},\ddot{q}) \tag{2.5-22}$$

and φ is the vector of arm parameters.

EXAMPLE 2.5-1: Cartesian Dynamics for Three-Link Cylindrical Arm

Let us show how to convert the joint space dynamics found in Example 2.2-3 to Cartesian dynamics. From Example A.3-1, the arm Jacobian is

$$J = \begin{bmatrix} -r\cos\theta & 0 & -\sin\theta \\ -r\cos\theta & 0 & \cos\theta \\ 0 & 1 & 0 \end{bmatrix}, \tag{1}$$

whence its inverse is

$$J^{-1} = \begin{bmatrix} -(\cos\theta)/r & -(\sin\theta)/r & 0 \\ 0 & 0 & 1 \\ -\sin\theta & \cos\theta & 0 \end{bmatrix}. \tag{2}$$

From Example 2.2-3 the arm inertia matrix is

$$M = \begin{bmatrix} J + m_2 r^2 & 0 & 0 \\ 0 & m_1 + m_2 & 0 \\ 0 & 0 & m_2 \end{bmatrix} \tag{3}$$

Applying (2.5-11) yields (verify!)

$$\overline{M} = J^{-T} M J^{-1} = \begin{bmatrix} m_2 + \overline{J}\cos^2\theta & \overline{J}\sin\theta\cos\theta & 0 \\ \overline{J}\sin\theta\cos\theta & m_2 + \overline{J}\sin^2\theta & 0 \\ 0 & 0 & m_1 + m_2 \end{bmatrix}. \tag{4}$$

where $\overline{J} \equiv J/r^2$.

In a similar fashion, one may compute $\overline{N}$.