Motion Equations of Mobile robots

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Abstract.- *In this resume is presented a model of movement for mobile robots [MR], the synchro-drive-and-steering-wheel vehicle, and its application to two configurations of mobile robots considering their kinematics constraints.*

1. Model

Here are considered polar and parametric representation of the path (z). In both cases the input is the speed (v), the angle of the speed (θ) and a function of the angular velocity (ω). The outputs are the states, the position of the robot, components of the velocity and the distance traveled along to the path.

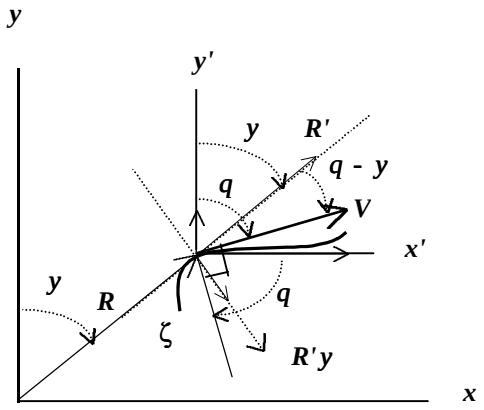


Figure 1.

State equations

From the figure and after some manipulations, we can obtain state equations

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}=\begin{bmatrix}\sin q\\ \cos q\end{bmatrix}V \qquad (1)$$

$$\begin{bmatrix}\dot{R}\\ \dot{\Psi}\end{bmatrix}=\begin{bmatrix}\cos(q-\Psi)\\ \dfrac{\sin(q-\Psi)}{R}\end{bmatrix}V \quad . \qquad (2)$$

The state-space transformation relating these two equations is given by:

$$\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}\sin\Psi\\ \cos\Psi\end{bmatrix}R \qquad (3)$$

$$\begin{bmatrix}R\\ \Psi\end{bmatrix}=\begin{bmatrix}\sqrt{x^2+y^2}\\ \tan^{-1}\left(x/y\right)\end{bmatrix}V \quad . \qquad (4)$$

Also we can relate the velocities according to

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}=\begin{bmatrix}S\Psi & C\Psi\\ C\Psi & -S\Psi\end{bmatrix}\begin{bmatrix}\dot{R}\\ R\dot{\Psi}\end{bmatrix} \qquad (5)$$

$$\begin{bmatrix}\dot{R}\\ R\dot{\Psi}\end{bmatrix}=\begin{bmatrix}S\Psi & C\Psi\\ C\Psi & -S\Psi\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} \qquad (6)$$

Nonholonomic constraint

From (6) it is also obtained

$$\begin{bmatrix}V\\ \mathbf{0}\end{bmatrix}=\begin{bmatrix}Sq & Cq\\ Cq & -Sq\end{bmatrix}\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} \qquad (7)$$

The bottom part of equation (7) defines the condition of zero motion sideways. This is a nonholonomic constraint.

Position motion variables

The distance traveled by the robot is defined in parametric and polar form by

$$s=\int_a^b\sqrt{\dot{x}^2+\dot{y}^2}\,dt \qquad (8)$$

$$s=\int_a^b\sqrt{\left(\frac{d(R)}{d\Psi}\right)^2+R^2}\,d\Psi \qquad (9)$$

Therefore

$$\dot{s} = V = r\dot{q} \quad . \quad \textbf{(10)}$$

The acceleration is defined by:

$$a(t) = a_N N + a_T T \quad \textbf{(11)}$$

$$a_N = \Re V^2 = \frac{V^2}{r} \quad \textbf{(12)}$$

$$a_T = \dot{V} \quad . \quad \textbf{(13)}$$

and the curvature and the radius of curvature are defined by

$$\Re = \frac{|v \times a|}{|v|^3} = \frac{|z'(t) \times z''(t)|}{|z'(t)|^3} \quad \textbf{(14)}$$

$$r = 1/\Re = \frac{1}{\frac{|dT/dt|}{ds/dt}} \quad \textbf{(15)}$$

with

$$z(t) = x(t)\hat{i} + y(t)\hat{j} \quad . \quad \textbf{(16)}$$

From (15), (16) and (17), considering that the robot is moving in a plane, the equations could be rewritten in the following way

$$\Re = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{\left(\sqrt{\dot{x}^2 + \dot{y}^2}\right)^3} \quad (\textbf{17})$$

$$r = \frac{\left(\sqrt{\dot{x}^2 + \dot{y}^2}\right)^3}{\dot{x}\ddot{y} - \ddot{x}\dot{y}} \quad . \quad (\textbf{18})$$

From (1), (17) and (18)

$$r = \frac{V}{|\dot{q}|} \quad \textbf{(19)}$$

$$\Re = \frac{|\dot{q}|}{V} \quad \textbf{(20)}$$

$$a_N = \frac{V^2}{r} = |\dot{q}|V \quad \textbf{(21)}$$

$$a_T = \dot{V} \quad . \quad \textbf{(22)}$$

The proof of these expressions is attached in appendix A.

2. Selection of control inputs: Vehicle kinematics

2.1 One front drive steering wheel.

We can get that,

$$V = V_s \cos f \quad \textbf{(23)}$$

is the velocity in the middle of the rear wheels (point p). Also at the same point,

$$\dot{x} = V \sin q = V_s \cos f \sin q \quad \textbf{(24)}$$

$$\dot{y} = V \cos q = V_s \cos f \cos q \quad \textbf{(25)}$$

$$\dot{q} = \frac{1}{l}\left[V_s \sin f + s\dot{f} \cos f\right] \quad \textbf{(26)}$$

$$\dot{s} = V = V_s \cos f \quad . \quad \textbf{(27)}$$

With control inputs v and f. Working with these equations, the curvature, radius of curvature, normal acceleration and tangential acceleration are defined in the following way,

$$\Re = \frac{1}{l}\left|\tan f + \frac{s\dot{f}}{V_s}\right| \quad (\textbf{28})$$

$$r = l\left|\frac{1}{\tan f + \frac{s\dot{f}}{V_s}}\right| \quad (\textbf{29})$$

$$a_N = \frac{V_s^2}{l}\left|\cos f\left(\sin f + \frac{s\dot{f}\cos f}{V_s}\right)\right| \quad (\textbf{30})$$

$$a_T = \dot{V}_s \cos f - V_s \dot{f} \sin f \quad . \quad (\textbf{31})$$

2.2 Two-rear-drive-wheel vehicle.

In the middle point of the rear wheels axe, the velocity is defined by,

$$V = \frac{1}{2}(V_1 + V_2) \qquad \textbf{(32)}$$

also at the same point,

$$\dot{x} = V \sin q = \frac{1}{2}(V_1 + V_2)\sin q \qquad \textbf{(33)}$$

$$\dot{y} = V \cos q = \frac{1}{2}(V_1 + V_2)\cos q \qquad \textbf{(34)}$$

$$\dot{q} = \frac{1}{l}\left[V_2 - V_1\right] \quad . \qquad \textbf{(35)}$$

With control inputs V_1 and V_2 working with these equations, it is obtained the following expressions,

$$\Re = \frac{2}{l}\left|\frac{V_2 - V_1}{V_1 + V_2}\right| \qquad \textbf{(36)}$$

$$r = \frac{l}{2}\left|\frac{V_1 + V_2}{V_2 - V_1}\right| \qquad \textbf{(37)}$$

$$a_N = \frac{1}{2l}\left|V_2 - V_1\right|(V_1 + V_2) \qquad \textbf{(38)}$$

$$a_T = \frac{1}{2}(\dot{V}_1 + \dot{V}_2) \quad . \qquad \textbf{(39)}$$

The proof of these expressions (section 2.1, 2.2) is attached in appendixes B and C.

3. References.

[1] J. C. Alexander and J. H. Maddocks, "On the Kinematics of Wheeled Mobile Robots", *Int. J. Robotics Research*, vol. 8, no. 5, pp. 15-27, Aug. 1989.

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