Thus (n,o,a) are the base coordinates of an (x,y,z) Cartesian coordinate system attached to the end effector. The *position* vector p specifies the location of the origin of the (n,o,a) frame with respect to the base frame.

The representation (n,o,a) for the orientation of the end effector is inefficient. Note that $[n \quad o \quad a]$ is a 3×3 matrix, so that it has nine entries. However, it does not take nine degress of freedom to specify orientation. Indeed, $[n \quad o \quad a]$ is a rotation matrix, so that its columns are orthogonal; this orthogonality requirement imposes extra constraints on the elements of $[n \quad o \quad a]$, so that the nine entries of $[n \quad o \quad a]$ are not independent. Alternative more efficient methods of specifying the orientation of the end effector in base coordinates are the roll–pitch–yaw, Euler angle, quaternion, and tool-configuration vector descriptions. We discuss some of these later. The reason we use (n,o,a) is that the arm T matrix is easily computed using the A matrices in terms of the arm parameters and join variables. Then n, o, a, and p are simply read off by examining the T matrix, as we shall see in the examples.

At this point we should like to distinguish between two sets of coordinates describing the end effector. The *joint variable* coordinates of the end effector are given by the n-vector.

$$q = [q_1 \quad q_2 \quad \cdots \quad q_n]^T,$$

where q_i can represent angles or lengths, depending on whether the links are revolute or prismatic. Generally, $n = 6$, so that the arm has six degrees of freedom. The end-effector *Cartesian coordinates* are the description of end-effector orientation and position in terms of the arm base coordinates. According to (A.2-11), where T may be computed knowing q, the joint variable and Cartesian coordinates are equivalent, for both specify the location of the end effector.

We say that q is the *joint-space* description of the position and orientation of the end effector, while (n,o,a,p) is the *Cartesian* or *task space* description. This terminology derives from the fact that descriptions of tasks to be performed by an arm are generally given in Cartesian coordinates, not in joint coordinates.

The robot arm *kinematics problem* is as follows. Given the joint variables q, find the Cartesian position and orientation of the end of the manipulator. Thus the kinematics problem amounts to converting given joint variables into the Cartesian position and orientation of the end effector expressed in base coordinates. Let us illustrate the solution of this problem for several simple robot arms which we use as examples throughout the book. It amounts to computing T.

It should be mentioned that there are several software packages commercially available for computing the A matrices and the T matrix for a given robot arm. See, for instance, [MATMAN 1986].

EXAMPLE A.2-1 Kinematics for Three-Link Cylindrical Arm

A simple RPP manipulator is shown in Fig. A.2-3a. It may be interpreted as the first three joints of an arm much as the GMF M-100. These are the joints used to position the end effector. A wrist mechanism consisting of three joints may be added to the end of the RPP arm to orient the end effector in space (see Example A.2-4). The joint variables of the three-link arm shown are θ, h, r, which correspond to the coordinates of a cylindrical coordinate system, so that the joint-variable vector is

$$q = [\theta \quad h \quad r]^T. \tag{1}$$

a. A Matrices

Coordinate frames may be attached to links 1, 2, and 3 using any technique desired. The D-H frames, to which (A.2.1) corresponds, are shown in Fig. A.2-3b. For this choice of frames, the A matrices may be determined as follows.

Frame 1 is related to the base frame 0 by a simple rotation of θ degrees about the axis z_0. A z-axis rotation, $R_{z,\theta}$, with no translation is described by

$$A_1 = \left[\begin{array}{ccc|c} c\theta & -s\theta & 0 & 0 \\ s\theta & c\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right], \tag{2}$$

where $c\theta$ represents $\cos\theta$ and $s\theta$ represents $\sin\theta$.

To find A_2 we may use (A.2-1). Since link 2 is prismatic with extension h, the length a_2 is zero. In this example the rotation θ_2 is also zero. The twist α_2 of link 2 is the angle of rotation about axis x_2 required to align z_1 with z_2 – that is, $-90°$.

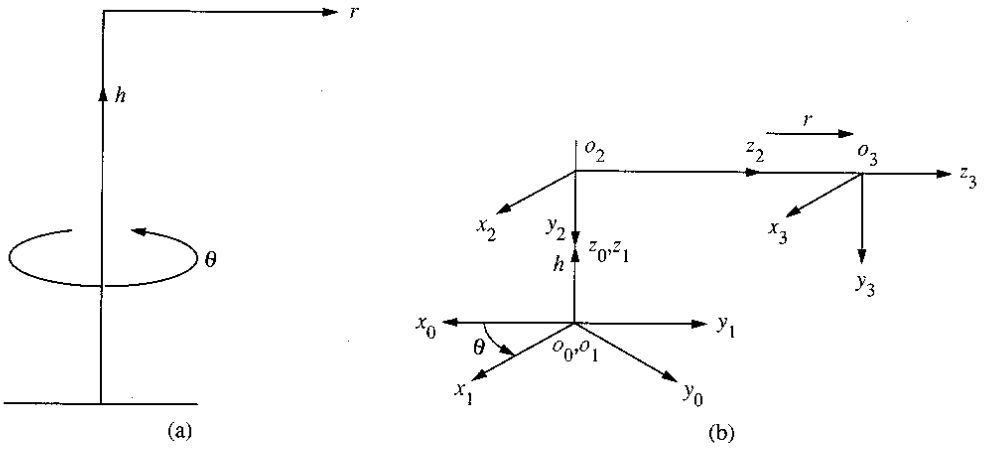


FIGURE A.2-3 Three-link cylindrical manipulator: (a) arm schematic; (b) D-H coordinate frames.

Therefore,

$$A_2 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & h \\ \hline 0 & 0 & 0 & 1 \end{array}\right]. \tag{3}$$

There is an attractive alternative to (A.2.1) for determining the link A matrix in simple cases. Recall that the first column of A_2 is the representation of x_2 in the coordinates (x_1,y_1,z_1). According to the figure, this is just $[1 \quad 0 \quad 0]^T$. The second column of A_2 is the representation of y_2 in the coordinates (x_1,y_1,z_1), which the figure shows is $[0 \quad 0 \quad -1]^T$. The third column of A_2 is the representation of z_2 in frame o_1, which is just $[0 \quad 1 \quad 0]^T$. Thus (3) is obtained by inspection.

In similar fashion we obtain,

$$A_3 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & r \\ \hline 0 & 0 & 0 & 1 \end{array}\right]. \tag{4}$$

b. T Matrix and Arm Kinematics

The arm T matrix is now obtained as

$$T = A_1A_2A_3 = \left[\begin{array}{ccc|c} c\theta & 0 & -s\theta & -r\,s\theta \\ s\theta & 0 & c\theta & r\,c\theta \\ 0 & -1 & 0 & h \\ \hline 0 & 0 & 0 & 1 \end{array}\right]. \tag{5}$$

To interpret the T matrix, examine (A.2-11). The position in base coordinates of the end of the manipulator, that is, the origin of frame 3, is

$$p = [-r\,s\theta \quad r\,c\theta \quad h]^T. \tag{6}$$

The orientation of frame 3 described in base coordinates is expressed in (n,o,a) form by giving the coordinates of the normal, orientation, and approach vectors in terms of the base frame. These are given by

$$\begin{aligned} n &= [c\theta \quad s\theta \quad 0]^T \\ o &= [0 \quad 0 \quad -1]^T \\ a &= [-s\theta \quad c\theta \quad 0]^T. \end{aligned} \tag{7}$$

A glance at Fig. A.2.3b verifies these expressions.

EXAMPLE A.2-2 Kinematics for Two-Link Planar Elbow Arm

A two-link planar RR arm is shown in Fig. A.2-4a where a_1 and a_2 are the fixed and known link length parameters. The link coordinate frames are shown in Fig. A.2-4b. We have taken the z-axes perpendicular to the page to conform to the con-

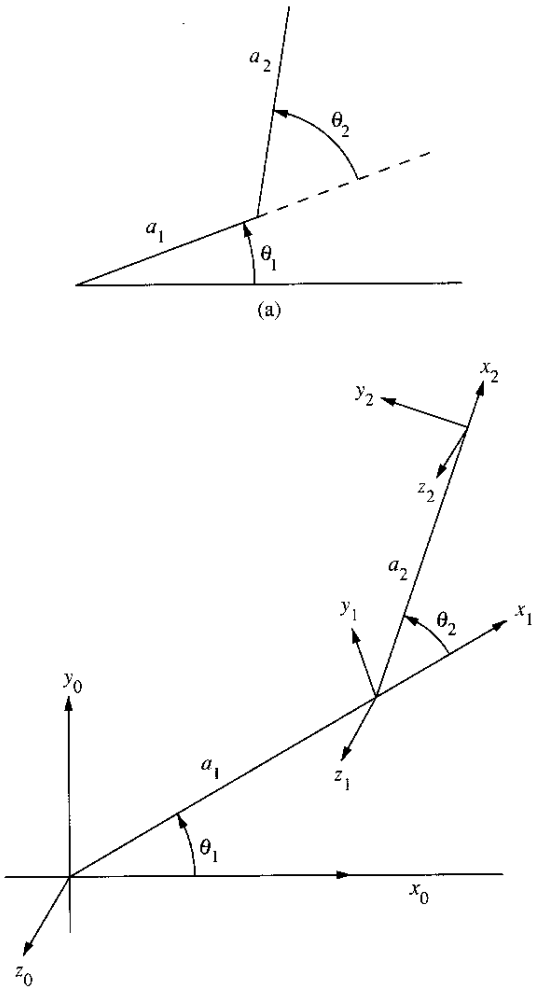


FIGURE A.2-4 Two-link planar RR arm: (a) arm schematic; (b) D-H coordinate frames.

vention of specifying points in a plane by (x,y) coordinates. Therefore, the frames are not defined quite as in the D-H convention.

The arm A matrices may be written by inspection as

$$A_1 = \left[\begin{array}{ccc|c} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]. \tag{1}$$

where c_1, s_1 represent respectively $\cos\theta_1$, $\sin\theta_1$, and

$$A_2 = \left[\begin{array}{ccc|c} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]. \tag{2}$$

The arm T matrix is given by

$$T = A_1A_2 = \left[\begin{array}{ccc|c} c_{12} & -s_{12} & 0 & a_1c_1+a_2c_{12} \\ s_{12} & c_{12} & 0 & a_1s_1+a_2s_{12} \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]. \tag{3}$$

where $c_{12} \equiv \cos(\theta_1 + \theta_2)$ and $s_{12} \equiv \sin(\theta_1 + \theta_2)$.

Therefore, the origin o_2 of frame 2 in terms of base coordinates is located at

$$p = [a_1c_1+a_2c_{12} \quad a_1s_1+a_2s_{12} \quad 0]^T. \tag{4}$$

This represents the kinematic solution, which converts the joint variable coordinates (θ_1,θ_2) into the base-frame Cartesian coordinates of the end of the arm. The reader should examine Fig. A.2.4a to verify this expression.

EXAMPLE A.2-3 Kinematics for Two-Link Polar Arm

A two-link planar RP arm is shown in Fig. A.2-5a where ℓ is the fixed known length of the base link. The link frames, with the z–axes perpendicular to the page, are shown in Fig. A.2-5b. The joint vector is

$$q = [\theta \quad r]^T, \tag{1}$$

which corresponds to polar coordinates in the plane.

By inspection, the A matrices are found to be

$$A_1 = \left[\begin{array}{ccc|c} c\theta & -s\theta & 0 & 0 \\ s\theta & c\theta & 0 & \ell \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 1 \end{array}\right] \tag{2}$$

$$A_2 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]. \tag{3}$$

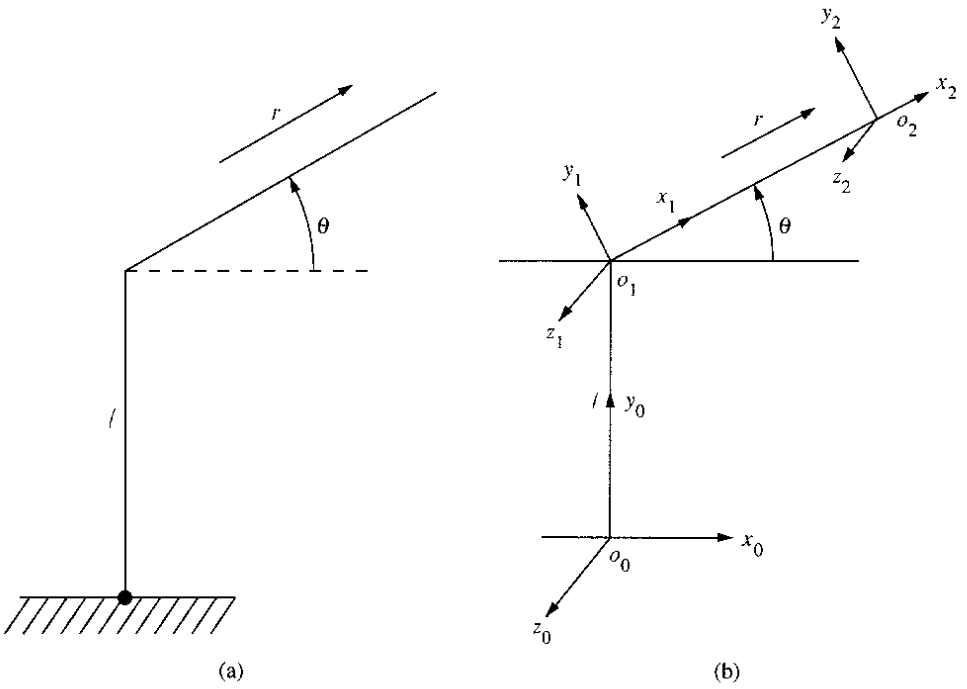


FIGURE A.2-5 Two-link planar RP arm: (a) arm schematic; (b) D-H coordinate frames.

The T matrix is

$$T = A_1A_2 = \left[\begin{array}{ccc|c} c\theta & -s\theta & 0 & rc\theta \\ s\theta & c\theta & 0 & \ell+rs\theta \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]. \tag{4}$$

Therefore, the base coordinates of the end of the arm are

$$p = [r\ c\theta \quad \ell+rs\theta \quad 0]^T, \tag{5}$$

which should be verified by examining Fig. A.2-5a. This may be interpreted as the forward kinematics solution for the arm.

This example illustrates the freedom we have to select the base frame origin o_0. We could have selected o_0 coincident with o_1. However, we have chosen to include the length ℓ of link 0 by placing o_0 at the bottom of the base link.

EXAMPLE A.2-4 Kinematics for Spherical Wrist

A spherical wrist mechanism is shown in Fig. A.2-6. See [Paul 1981, Spong and Vidyasagar 1989]. For convenient orientational control, all three joint axes intersect at a common point. Since many six-link industrial arms, including the Stanford arm, end in such a configuration, we have labeled the three joint varia-

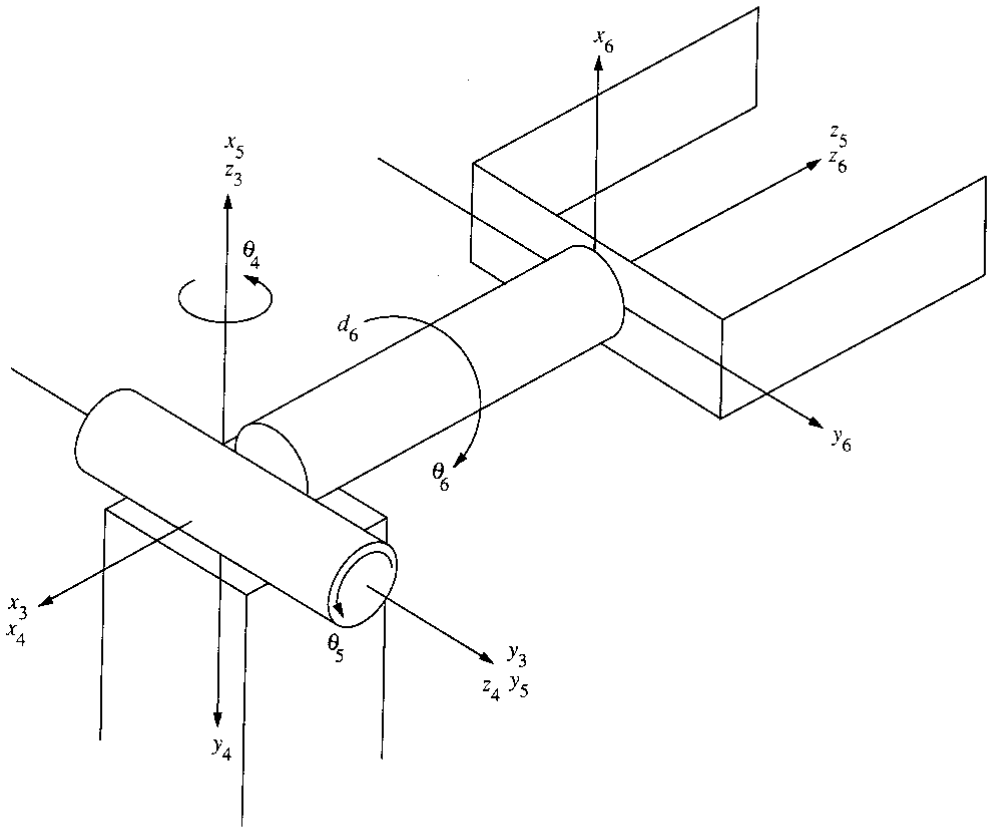


FIGURE A.2-6 Spherical wrist.

bles θ_4, θ_5, θ_6. The D-H frames are also shown in the figure. Recall that the rotation θ_i occurs about axis z_{i-1} in the D-H convention. The origin of frame 6 has been chosen at the base of the fingered gripper. The length d_6 is a fixed known parameter.

By expressing the axes x_4, y_4, z_4 in terms of the coordinates (x_3,y_3,z_3), we are able to directly determine the columns of A_4 to obtain

$$A_4 = \left[\begin{array}{ccc|c} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]. \tag{1}$$

Determining x_5, y_5, z_5 in terms of (x_4,y_4,z_4) yields

$$A_5 = \left[\begin{array}{ccc|c} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]. \tag{2}$$

In writing down A_5, it is important to note that θ_5 is defined to be zero when the end effector is in an upright position, that is, when the arm is fully extended. We have drawn the figure with $\theta_5 = -90°$ to show more clearly the different coordinate systems.

In similar fashion, we obtain

$$A_6 = \left[\begin{array}{ccc|c} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]. \tag{3}$$

The wrist T matrix is

$$T = A_4A_5A_6 = \left[\begin{array}{ccc|c} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 & c_4s_5d_6 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 & s_4s_5d_6 \\ -s_5c_6 & s_5s_6 & c_5 & c_5d_6 \\ \hline 0 & 0 & 0 & 1 \end{array}\right], \tag{3}$$

where $c_4 \equiv \cos\theta_4$, and so on. It is quite interesting to note that the rotational part of this T matrix corresponds to the Euler angle transformation [Spong and Vidyasagar 1989]. Therefore, θ_4, θ_5, θ_6 may be interpreted as the Euler angles with respect to frame 3.

Suppose that we would like to determine the kinematic transformations of the 3-link cylindrical arm in Example A.2-1 terminated with a spherical wrist. Then it is only necessary to multiply the T matrix in Example A.2-1 and (4), in that order, to obtain the overall manipulator T matrix.

Inverse Kinematics

The location of the end effector is specified in base coordinates by the arm T matrix. On the other hand, the location of the end effector is specified in joint coordinates by giving the values of the joint variables q_i. We may compute the T matrix knowing the joint-variable vector q, as in the examples just shown. Finding T from q is the kinematics problem.

The *inverse-kinematics* problem is as follows. Given (n,o,a,p) for the end effector in base coordinates, determine the joint variables q_i in the T matrix

$$T = \begin{bmatrix} n & o & a & p \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{A.2-12}$$

that yield the specified (n,o,a,p). This problem is important in manipulator control, for the desired Cartesian orientation and position of the end effector are specified by the task. Then the solution to the inverse kinematics problems gives the joint variables q_i required to achieve that orientation and position.

Due to the functions involved in the T matrix, the relations between q_i and (n,o,a,p) are highly nonlinear; these must be inverted to obtain the inverse kinematic solution q. The solutions for q_i are generally not unique. If the last three arm axes intersect at a point (e.g., the wrist mechanism in Example A.2-4), it is common to split the inverse kinematics problem into two parts. In a six-link arm, for instance, one first determines q_1, q_2, and q_3 required to obtain the desired Cartesian position p. Then the wrist variables q_4, q_5, and q_6 that give the desired orientation (n,o,a) are determined. Some techniques for solving the inverse kinematics problem are given in [Paul 1981, Craig 1989, Spong and Vidyasagar 1989]. An example is now given.

EXAMPLE A.2-5 Inverse Kinematics for Two-Link Planar Elbow Arm

For the planar RR arm of Example A.2-2, the inverse kinematics problem amounts to finding the joint variable θ_1 and θ_2 given a desired Cartesian position (x,y) of the end of the arm. See Fig. A.2-7.

The first thing that is evident is that, as long as $a_1^2 + a_2^2 < r \equiv x^2 + y^2$, there are two solutions. The one shown in Fig. A.2-7 is the "elbow down" solution. Another solution may be determined for the "elbow up" configuration, where both links are above the vector $[x \quad y]^T$. Thus *the inverse kinematics problem generally has a nonunique solution.* This may often be taken advantage of to obtain end-effector positioning with *collision avoidance.*

Given the T matrix from Example A.2-2, we see that θ_1 and θ_2 may be found by solving

$$a_1 c_1 + a_2 c_{12} = x \tag{1}$$

$$a_1 s_1 + a_2 s_{12} = y. \tag{2}$$

Determining θ_1 and θ_2 based on algebraic manipulations of such equations is called the *algebraic* inverse kinematics solution technique [Paul 1981]. Let us show a *geometric technique* here.

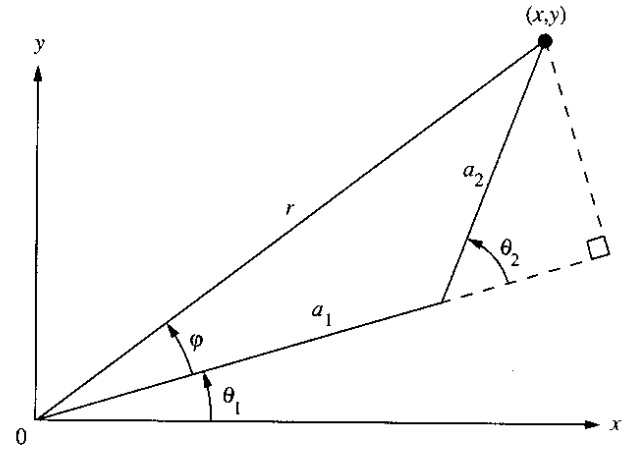


FIGURE A.2-7 Inverse kinematics for two-link planar RR arm.

Referring to Fig. A.2-7, define

$$r^2 = x^2 + y^2 \tag{3}$$

and use the law of cosines to obtain

$$\begin{aligned} r^2 &= a_1^2 + a_2^2 - 2a_1a_2 \cos(\pi - \theta_2) \\ &= a_1^2 + a_2^2 + 2a_1a_2 \cos\theta_2. \end{aligned} \tag{4}$$

We could now solve for θ_2 using the $\cos^{-1}$ function. However, it is better to use $\tan^{-1}$ for reasons of numerical accuracy. The FORTRAN function implementing $\tan^{-1}(b/c)$ is ATAN2(b,c). This function has a uniform accuracy over the range of its arguments, returns a unique value for the angle depending on the signs of b and c, and gives the correct solution if b and/or c is zero.

Therefore, we proceed by computing

$$\cos\theta_2 = \frac{r^2 - a_1^2 - a_2^2}{2a_1a_2} \equiv C \tag{5}$$

$$\sin\theta_2 = \pm\sqrt{1 - \cos^2\theta_2} = \pm\sqrt{1 - C^2} \equiv D \tag{6}$$

$$\theta_2 = \text{ATAN2(D,C)}. \tag{7}$$

An additional advantage of using the arctangent is that the multiple solutions of the inverse kinematics problem are explicitly revealed by the choice of negative or positive sign in (6).

To determine θ_1, define the auxiliary angle φ in the figure. By inspection of the right triangle shown,

$$\tan\varphi = \frac{a_2 \sin\theta_2}{a_1 + a_2\cos\theta_2}. \tag{8}$$

Moreover,

$$\tan(\varphi + \theta_1) = \frac{y}{x}, \tag{9}$$

so that

$$\theta_1 = \text{ATAN2(y,x)} - \text{ATAN2}(a_2 \sin\theta_2,\ a_1 + a_2\cos\theta_2). \tag{10}$$

Note that θ_1 depends on θ_2.

A.3 The Manipulator Jacobian

Given a generally nonlinear transformation from the joint variable $q(t) \in \mathbb{R}^n$ to $y(t) \in \mathbb{R}^p$,

$$y = h(q) \tag{A.3-1}$$

we define the Jacobian associated with $h(q)$ as

$$J(q) \equiv \frac{\partial \mathrm{h}(q)}{\partial q}. \tag{A.3-2}$$

As we have just seen, the Jacobian is useful in feedback linearization, and therefore in robot manipulator control. It is also the means by which we transform velocity, acceleration, and force between coordinate frames.

Transformation of Velocity and Acceleration

Since

$$\dot{y} = \frac{\partial h}{\partial q}\,\dot{q} = J\dot{q}, \tag{A.3-3}$$

the Jacobian allows us to transform velocity from joint space to "y-space." Let us discuss the special case where $\dot{y}(t)$ is the Cartesian velocity. Then $J(q)$ is called the *manipulator Jacobian.*

It is usual to define the generalized Cartesian velocity as

$$\dot{y} = \begin{bmatrix} v \\ \omega \end{bmatrix}, \tag{A.3-4}$$

with $v = [v_x \quad v_x \quad v_z]^T$ the linear velocity and $\omega = [\omega_x \quad \omega_y \quad \omega_z]^T$ the angular velocity. For instance, ω_x represents angular velocity about the x-axis. Thus $\dot{y}$ has six components and the arm Jacobian J is a $6 \times n$ matrix, with n the number of joints in the manipulator. If $n = 6$, the Jacobian is square. We shall soon show how to compute the manipulator Jacobian.

Using (A.3-3), we can obtain an expression for the transformation of *differential motion.* Let $dq = [dq_1 \cdots dq_n]^T$ be a differential motion in joint space, with dq_i a small rotation if joint i is revolute, and a small linear displacement if joint i is prismatic. Let

$$\mathbf{dy} = \begin{bmatrix} dx \\ dy \\ dz \\ \delta x \\ \delta y \\ \delta z \end{bmatrix} \tag{A.3-5}$$

describe the same differential motion in Cartesian coordinates, with $[d_x \quad d_y \quad d_z]^T$ the differential linear motion and $[\delta x \quad \delta y \quad \delta z]^T$ representing the differential rotation. Here δx, for instance, means a small rotation about the x-axis. We have written the Cartesian differential motion vector **dy** in boldface to distinguish it from the small change dy in the second coordinate of Cartesian position.

According to (A.3-3), where $\dot{y} = dy/dt$ and $\dot{q} = dq/dt$, we see that

$$\mathbf{dy} = J\, dq, \tag{A.3-6}$$

with J the Jacobian relating joint space and Cartesian space.

The transformation of acceleration is found by differentiating (A.3-3) to be

$$\ddot{y} = J\ddot{q} + \dot{J}\dot{q}. \tag{A.3-7}$$

Transformation of Force

To discover the transformation of static force between coordinate frames, consider the following.

The *virtual work* resulting from the application of a generalized force/torque in joint space that results in a differential motion dq is

$$\delta W = \tau^T dq, \tag{A.3-8}$$

with τ the n-vector of arm control torques/forces, and $dq = [dq_1 \cdots dq_n]^T$ the differential change in the joint variable. If the description in another coordinate frame of the force is F and the description there of position is y, we must also have

$$\delta W = F^T \mathbf{dy}. \tag{A.3-9}$$

Now taking (A.3-6) into account, we may write

$$\delta W = \tau^T dq = F^T J dq,$$

so that the transformation from force to torque is given by

$$\tau = J^T(q)F. \tag{A.3-10}$$

In the case that $y(t)$ is Cartesian position, we define the Cartesian generalized force to be the 6-vector

$$F = \begin{bmatrix} f_c \\ \tau_c \end{bmatrix}, \tag{A.3-11}$$

with $\mathrm{f_c} = [f_x \;\; f_y \;\; f_z]^T$ a Cartesian force 3-vector, and $\tau_c = [\tau_x \;\; \tau_y \;\; \tau_z]^T$ a 3-vector representing Cartesian torques. For instance, τ_x represents torque exerted about the x-axis.

If the arm has six links and the Jacobian is nonsingular, the transformation from generalized torque to generalized force is given by

$$F = J^{-T}(q)\tau. \tag{A.3-12}$$

The singularities in the Jacobian generally occur at the extremities of the manipulator workspace.