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EE 4343/5329 - Control System Design Project

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SOME REPRESENTATIVE DYNAMICAL SYSTEMS

We discuss modeling of dynamical systems. Several interesting systems are discussed that are representative of different classes of dynamics.

Modeling Physical Systems

The nonlinear state-space equation is

$$\dot{x} = f(x,u)$$
$$y = h(x,u)$$

with $x(t) \in R^n$ the internal state, $u(t) \in R^m$ the control input, and $y(t) \in R^p$ the measured output. If we can find a mathematical model of this form for a system, then computer simulation is very easy and feedback controller design is facilitated. To find the state equations for a given system, several techniques can be used. In electronic circuit analysis, for instance, KVL and KCL directly give the state-space form. An equivalent technique based on flow conservation is used in the analysis of hydraulic systems.

For the analysis of mechanical systems we can use Hamilton's equations of motion or Lagrange's equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = F,$$

with $q(t)$ the generalized position vector, $\dot{q}(t)$ the generalized velocity vector, and F(t) the generalized force vector. The Lagrangian is L= K-U, the kinetic energy minus the potential energy.

The linear state-space equations are given by

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

where A is the system or plant matrix, B is the control input matrix, C is the output or measurement matrix, and D is the direct feed matrix. This linear form is very convenient for the design of feedback control systems.

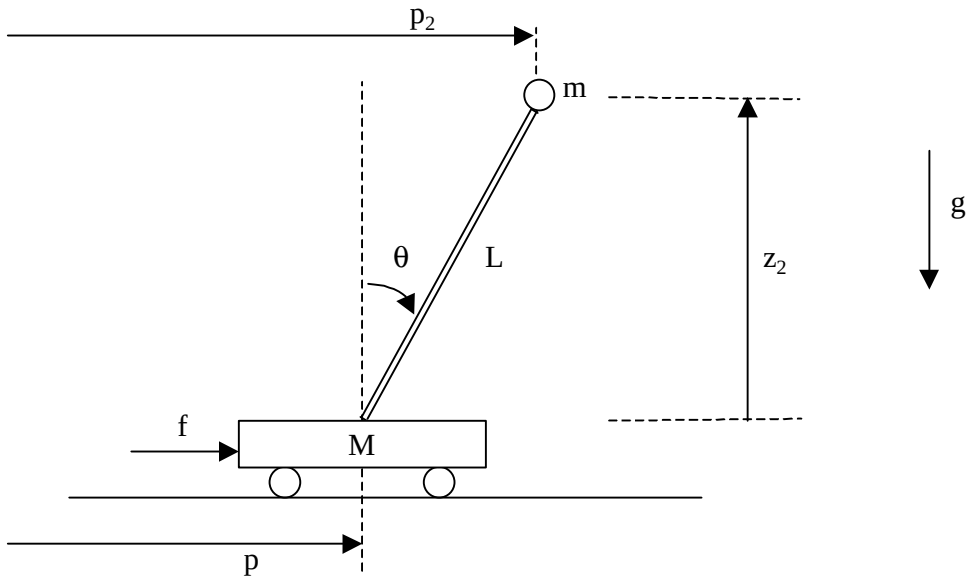
The linear state-space form is obtained directly from a physical analysis if the system is inherently linear. If the system is nonlinear, then the state equations are nonlinear. In this case, an approximate linearized system description may be obtained by computing the Jacobian matrices

$$A(x,u) = \frac{\partial f}{\partial x}, \quad B(x,u) = \frac{\partial f}{\partial u}, \quad C(x,u) = \frac{\partial h}{\partial x}, \quad D(x,u) = \frac{\partial h}{\partial u}.$$

These are evaluated at a nominal set point (x,u) to obtain constant system matrices A,B,C,D, yielding a linear time-invariant state description which is approximately valid for small excursions about the nominal point.

INVERTED PENDULUM

The inverted pendulum on a cart is representative of a class of systems that includes stabilization of a rocket during launch, etc.. The position of the cart is p, the angle of the rod is θ, the force input to the cart is f, the cart mass is M, the mass of the bob is m, and the length of the rod is L. The coordinates of the bob are (p_2,z_2).



Inverted Pendulum

We want to use Lagrange's equation. The kinetic energy of the cart is

$$K_1 = \frac{1}{2} M \dot{p}^2 .$$

The kinetic energy of the bob is

$$K_2 = \frac{1}{2} m(\dot{p}_2^2 + \dot{z}_2^2)$$

where

$$p_2 = p + L \sin q, \quad z_2 = L \cos q$$

so that

$$\dot{p}_2 = \dot{p} + L\dot{q} \cos q, \quad \dot{z}_2 = -L\dot{q} \sin q .$$

Therefore, the total kinetic energy is

$$K = K_1 + K_2 = \frac{1}{2} M \dot{p}^2 + \frac{1}{2} m(\dot{p}^2 + 2\dot{p}\dot{q} L \cos q + L^2 \dot{q}^2) .$$

The potential energy is due to the bob and is

$$U = mgz_2 = mgL \cos q .$$

The Lagrangian is

$$L = K - U = \frac{1}{2}(M + m)\dot{p}^2 + mL\dot{p}\dot{q} \cos q + \frac{1}{2} mL^2 \dot{q}^2 - mgL \cos q .$$

The generalized coordinates are selected as $q = [q_1 \quad q_2]^T = [p \quad q]^T$ so that Lagrange's equations are

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{p}} - \frac{\partial L}{\partial p} &= f \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} &= 0 \end{aligned} .$$

Substituting for L and performing the partial differentiation yields

$$\begin{aligned} (M + m)\ddot{p} + mL\ddot{q} \cos q - mL\dot{q}^2 \sin q &= f \\ mL\ddot{p} \cos q + mL^2 \ddot{q} - mgL \sin q &= 0 \end{aligned} .$$

These dynamical equations must now be placed into state-space form. To accomplish this, write the Lagrange equation in terms of matrices as

$$\begin{bmatrix} M + m & mL \cos q \\ mL \cos q & mL^2 \end{bmatrix} \begin{bmatrix} \ddot{p} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} mL\dot{q}^2 \sin q + f \\ mgL \sin q \end{bmatrix} .$$

This is a mechanical system in typical Lagrangian form, with the inertia matrix multiplying the acceleration vector. The term $mL\dot{q}^2 \sin q$ is a centripetal term and $mgL \sin q$ is a gravity term.

Invert the inertia matrix and simplify to obtain

$$\ddot{p} = \frac{mg \sin q \cos q - mL\dot{q}^2 \sin q - f}{m \cos^2 q - (M+m)}$$

$$\ddot{q} = \frac{-(M+m)g \sin q + mL\dot{q}^2 \sin q \cos q + f \cos q}{mL \cos^2 q - (M+m)L}$$

Now, the state may be defined as $x = [x_1 \quad x_2 \quad x_3 \quad x_4]^T = [p \quad \dot{p} \quad q \quad \dot{q}]^T$ and the input as u= f. Then the nonlinear state equation may be written as

$$\dot{x} = \begin{bmatrix} x_2 \\ \dfrac{mg \sin x_3 \cos x_3 - mLx_4^2 \sin x_3 - u}{m \cos^2 x_3 - (M+m)} \\ x_4 \\ \dfrac{-(M+m)g \sin x_3 + mLx_4^2 \sin x_3 \cos x_3 + u \cos x_3}{mL \cos^2 x_3 - (M+m)L} \end{bmatrix} = f(x,u).$$

Given this nonlinear state equation, it is very easy to simulate the inverted pendulum behavior on a digital computer.

We now want to linearize this and obtain the linear state equation. The nominal point is x= 0, where the rod is upright. One could find Jacobians, but it is easier to use the approximations, valid near the origin, $\sin x_3 \approx x_3, \cos x_3 \approx 1$. In addition, all squared state components are very small and so set equal to zero. This yields the linear state equation

$$\dot{x} = \begin{bmatrix} x_2 \\ \dfrac{-mgx_3 + u}{M} \\ x_4 \\ \dfrac{(M+m)gx_3 - u}{ML} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{-mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{(M+m)g}{ML} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \dfrac{1}{M} \\ 0 \\ \dfrac{-1}{ML} \end{bmatrix} u = Ax + Bu.$$

The output equation depends on the measurements taken, which depends on the sensors available. Assuming measurements of cart position and rod angle, the output equation is

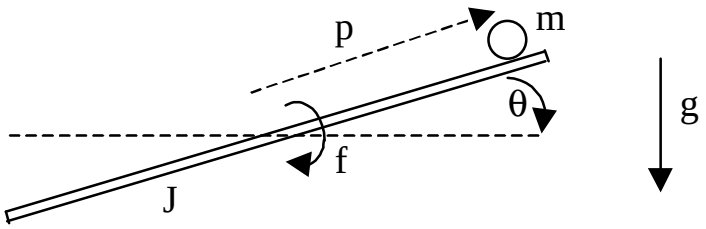
$$y = \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x = h(x,u).$$

The cart position may be measured by placing an optical encoder on one of the wheels, and the rod angle by placing an encoder at the rod pivot point. It is difficult to measure the velocities $\dot{p}, \dot{q}$, but this might be achieved by placing tachometers on a wheel and at the rod pivot point. Then, the output equation will change.

Given the linear state-space equations, a controller can be designed to keep the rod upright. Though the controller is designed using the linear state equations, the performance of the controller should be simulated in a closed-loop system using the full nonlinear dynamics $\dot{x} = f(x,u)$.

BALL BALANCER

The inverted pendulum can be viewed as a two-degrees-of-freedom robot arm with a prismatic (e.g. extensible) joint followed by a revolute (e.g. rotational) joint. It has only one actuator-- on the prismatic link. The ball balancing on a pivoted beam can be viewed as a robot arm with a revolute link followed by a prismatic link, also having only one actuator-- on the revolute link. This is in some sense a dual system to the inverted pendulum. The ball balancer is representative of a large class of systems in industrial and military applications. The position of the ball is p, the angle of the beam is θ, the torque input to the beam is f, the inertia of the beam is J, and the mass of the ball is m.



Ball Balancer

By finding the kinetic and potential energies and using Lagrange's equation, exactly as for the inverted pendulum, one determines that

$$\ddot{p} = p\dot{q}^2 - g\sin q$$

$$\ddot{q} = \frac{-2mp\dot{p}\dot{q} - mgp\cos q + f}{mp^2 + J}\cdot$$

Using these, the nonlinear state equations are easy to write down. The state may be selected as $x = [x_1 \quad x_2 \quad x_3 \quad x_4]^T = [p \quad \dot{p} \quad q \quad \dot{q}]^T$ and the input as u= f.

Selecting the nominal point as $p = p_n$ the desired ball position, $\dot{p} = 0, q = 0, \dot{q} = 0,$ one may use the approximations, valid near the origin, $\sin x_3 \approx x_3, \cos x_3 \approx 1$. In addition, all products of state components are very small and so are set equal to zero. This yields the linear state equation

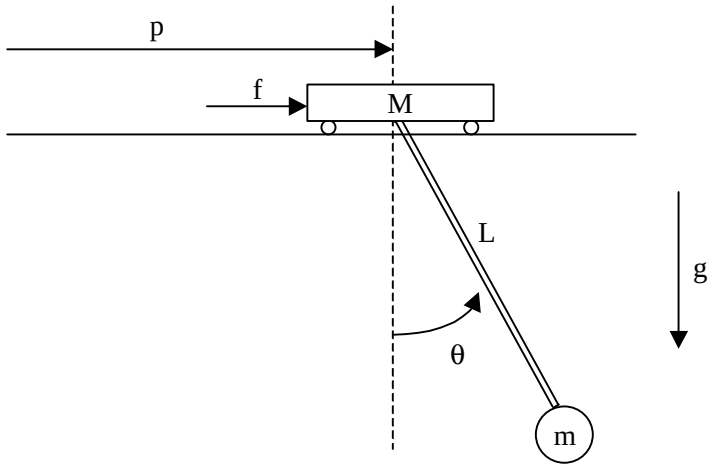
$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -g & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{-mg}{mp_n^{\,2} + J} & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{mp_n^{\,2} + J} \end{bmatrix} u = Ax + Bu \,.$$

The output equation depends on the measurements taken, which depends on the sensors available. Assuming measurements of ball position and beam angle, the output equation is

$$y = \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x = h(x,u) \,.$$

GANTRY CRANE

The gantry crane is a load suspended by a wire rope from a moving trolley. The horizontal position of the load is p, the angle of the wire is θ, the force input to the trolley is f, the mass of the trolley is M, and the mass of the load is m, and the length of the wire rope is L. Assume that the wire rope is stiff so that it does not flex or bend.



Gantry Crane

By finding the kinetic and potential energies and using Lagrange's equation, exactly as for the inverted pendulum, one determines that

$$\ddot{p} = \frac{-Mg\sin q\cos q - ML\dot{q}^2\sin q + f\sin^2 q}{m\sin^2 q + M}$$

$$\ddot{q} = \frac{-(M+m)g\sin q - mL\dot{q}^2\sin q\cos q - f\cos q}{mL\sin^2 q + ML}$$

Using these, the nonlinear state equations are easy to write down. The state may be selected as $x = [x_1 \quad x_2 \quad x_3 \quad x_4]^T = [p \quad \dot{p} \quad q \quad \dot{q}]^T$ and the input as u= f.

Selecting the nominal point as $p = p_n$ the desired load position, $\dot{p} = 0, q = 0, \dot{q} = 0,$ one may use the approximations, valid near the origin, $\sin x_3 \approx x_3, \cos x_3 \approx 1$. In addition, all products of state components are very small and so are set equal to zero. This yields the linear state equation

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-(M+m)g}{ML} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{-1}{ML} \end{bmatrix} u = Ax + Bu\,.$$

The output equation depends on the measurements taken, which depends on the sensors available. Assuming measurements of load position and wire angle, the output equation is

$$y = \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x = h(x,u)\,.$$

On the other hand, if the trolley position is measured, then one has the nonlinear output equation

$$y = p - L\sin q\,.$$

Linearizing this equation yields

$$y = [1 \quad 0 \quad -L \quad 0]x = h(x,u)\,.$$

MOTOR WITH COMPLIANT COUPLING

Motor drives with compliant coupling to a load occur throughout industrial applications. Also in this class are flexible-joint robot arms, where the actuators are coupled to the robot arm links through joint gearing which has some compliance.

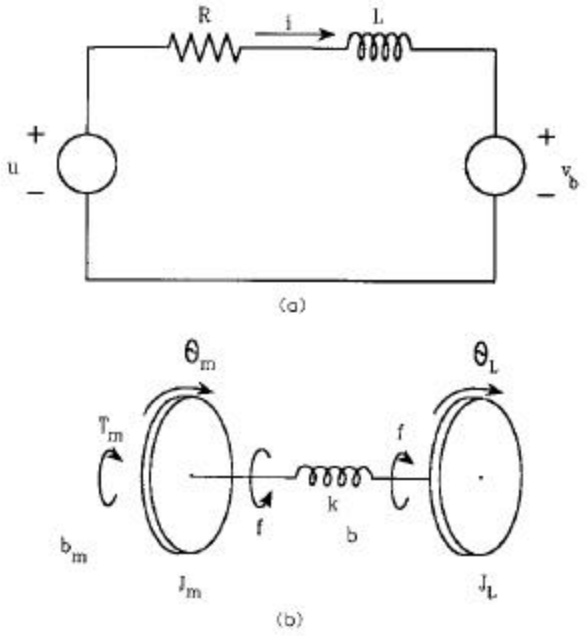


Figure 2.1-4 DC motor with shaft compliance. (a) Electrical subsystem. (b) Mechanical subsystem.

The mechanical equations are found to be of the form

$$J_m\ddot{q}_m + b_m\dot{q}_m + b(\dot{q}_m - \dot{q}_L) + k(q_m - q_L) = k_m i$$
$$J_L\ddot{q}_L - b(\dot{q}_m - \dot{q}_L) - k(q_m - q_L) = 0$$

where subscript 'm' refers to the motor, subscript 'L' refers to the load, J is inertia, b_m is the rotor equivalent damping constant, k_m is the motor torque constant, and the armature current i functions as a control input to the mechanical subsystem. The coupling shaft has spring constant k and damping b. The electrical subsystem dynamics must also be taken into account. The dynamics for an armature-coupled DC motor are described by

$$L\dot{i} + Ri + k_m'\dot{q}_m = u\,,$$

with L the armature winding leakage inductance, R the armature resistance, k_m' the back emf constant, and control input u(t) the armature voltage. The system is linear.

Selecting the state as $x = [x_1\ x_2\ x_3\ x_4\ x_5]^T = [i\ q_m\ w_m\ q_L\ w_L]^T$, with $w = \dot{q}$ the angular velocity, one may write the state equation as

$$\dot{x} = \begin{bmatrix} \frac{-R}{L} & 0 & \frac{-k_m'}{L} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{k_m}{J_m} & \frac{-k}{J_m} & \frac{-(b+b_m)}{J_m} & \frac{k}{J_m} & \frac{b}{J_m} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{k}{J_L} & \frac{b}{J_L} & \frac{-k}{J_L} & \frac{-b}{J_L} \end{bmatrix} x + \begin{bmatrix} \frac{1}{L} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u .$$

Assuming that the output of interest is the load angle, one has the output equation

$$y = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix} x = Cx .$$

Rigid Coupling Shaft

Adding the two mechanical subsystem equations together yields

$$J_m \ddot{q}_m + J_L \ddot{q}_L + b_m \dot{q}_m = k_m i .$$

If the coupling shaft is rigid, then $k = \infty, q_L = q_m$. Thus, the mechanical subsystem becomes

$$(J_m + J_L)\ddot{q}_m + b_m \dot{q}_m = k_m i .$$

Defining the total moment of inertia as $J = J_m + J_L$ and the state as $x = \begin{bmatrix} i & w_m \end{bmatrix}^T$ one now has the state equation

$$\dot{x} = \begin{bmatrix} -R/L & -k_m'/L \\ k_m/J & -b_m/J \end{bmatrix} x + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} u .$$

The following simulation is taken from F.L. Lewis, *Applied Optimal Control and Estimation*, Prentice-Hall, New Jersey, 1992 (copyright held by F.L. Lewis)

Example 2.1-8: Simulation of DC Motor with Compliant Coupling

In Example 2.1-3 we derived the state equations for an armature-controlled DC motor with a flexible coupling shaft. In this example we intend to use computer simulation to study the effects of the shaft compliance on the motor performance.

a. Rigid Coupling Shaft

If there is no compliance in the coupling shaft, the state equations are (see Example 2.1-3d)

$$\dot{x} = \begin{bmatrix} -R/L & -k_m'/L \\ k_m/J & -b_m/J \end{bmatrix} x + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} u \equiv Ax + Bu, \tag{1}$$

where $J = J_m + J_L$. Defining the output as the motor speed gives

$$y = [0 \quad 1]x \equiv Cx.$$

The state is $x = [i \quad \omega]^T$.

The transfer function is computed to be

$$H(s) = C(sI - A)^{-1}B = \frac{k_m}{(Ls + R)(Js + b_m) + k_m k_m'}. \tag{2}$$

Using parameter values of $J_m = J_L = 0.1$ kg.m^2, $k_m = k_m' = 1$ V.s, $L = 0.5$ H, $b_m = 0.2$ $N.m./\text{rad}/s$, and $R = 5\ \Omega$ yields

$$H(s) = \frac{10}{(s + 2.3)(s + 8.7)}, \tag{3}$$

so that there are two real poles at $s = -2.3$, $s = -8.7$.

Using Program TRESP in Appendix A to perform a simulation yields the step response shown in Fig. 2.1-10.

b. Very Flexible Coupling Shaft

Coupling shaft parameters of $k = 2$ $N.m./\text{rad}$ and $b = 0.2$ $N.m./\text{rad}/s$ correspond to a very flexible shaft. Using these values, software like PC-MATLAB [Moler et al. 1987] can be employed with the state model of Example 2.1-3c to obtain the two transfer functions

$$\frac{\omega_m}{u} = \frac{20s[(s + 1)^2 + 4.36^2]}{s(s + 3.05)(s + 6.14)[(s + 3.4)^2 + 5.6^2]} \tag{4}$$

$$\frac{\omega_L}{u} = \frac{40s(s + 10)}{s(s + 3.05)(s + 6.14)[(s + 3.4)^2 + 5.6^2]}. \tag{5}$$

The shaft flexible mode has the poles $s = -3.4 \pm j5.6$, and so has a damping ration of $\zeta = 0.52$ and a natural frequency of $\omega = 6.55$ rad/sec.

Note that the system is marginally stable, with a pole at $s = 0$. It is BIBO stable due to pole/zero cancellation.

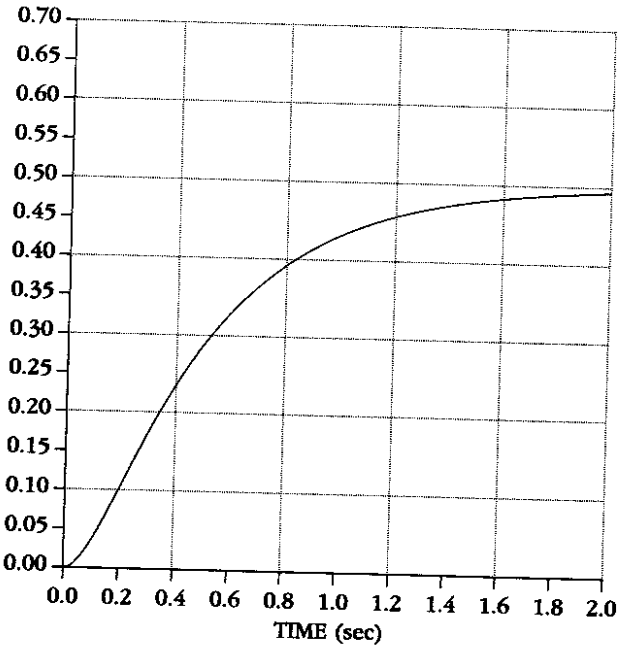


Figure 2.1-10 Step response of DC motor with no shaft flexibility. Motor speed in rad/sec.

Program TRESP yielded the step response shown in Fig. 2.1-11. Several points are worthy of note. Initially the motor speed ω_m rises more quickly than in Fig. 2.1-10, since the shaft flexibility means that only the rotor moment of inertia J_m initially affects the speed. Then, as the load J_L is coupled back to the motor through the shaft, the rate of increase of ω_m slows. Note also that the load speed ω_L exhibits a *delay* of approximately 0.1 sec due to the flexibility in the shaft.

It is extremely interesting to note that the shaft flexibility has the effect of speeding up the slowest motor real pole [compare (3) and (4)], so that ω_L approaches its steady-state value more quickly than in the rigid shaft case. This is due to the "whipping" action of the flexible shaft.

c. Fairly Rigid Coupling Shaft

Coupling shaft parameters of $k = 10$ *N.m.*/rad and $b = 1$ *N.m.*/rad/*s* correspond to a fairly rigid shaft. Using these values, the transfer functions from armature voltage to the motor and load speeds are

$$\frac{\omega_m}{u} = \frac{20s[(s + 5)^2 + 8.66^2]}{s(s + 2.4)(s + 7.43)[(s + 11.1^2) + 10.1^2]} \tag{6}$$

$$\frac{\omega_L}{u} = \frac{20s(s + 10)}{s(s + 2.4)(s + 7.43)[(s + 11.1^2) + 10.1^2]}. \tag{7}$$

The shaft flexible mode now occurs with poles at $s = -11.1 \pm j10.1$, and so has a damping ration of $\zeta = 0.74$ and a natural frequency of $\omega = 15$ rad/sec. Note that the real poles are similar to the rigid poles in (3).

Program TRESP yielded the step response shown in Fig. 2.1-12. This is much like the rigid response in Fig. 2.1-10, though the load speed ω_L still exhibits some initial delay.

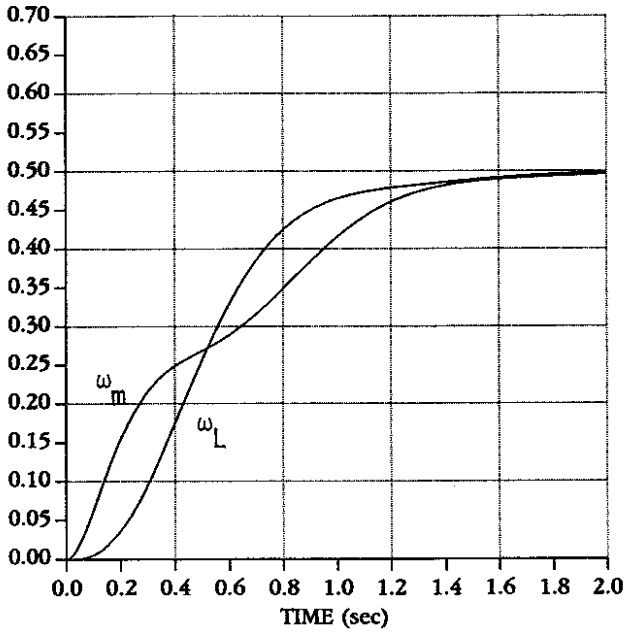


Figure 2.1-11 Step response of motor with very flexible shaft

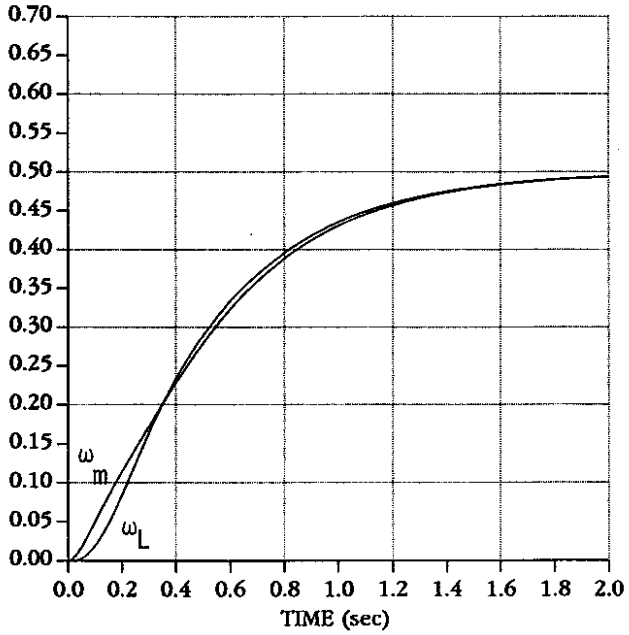
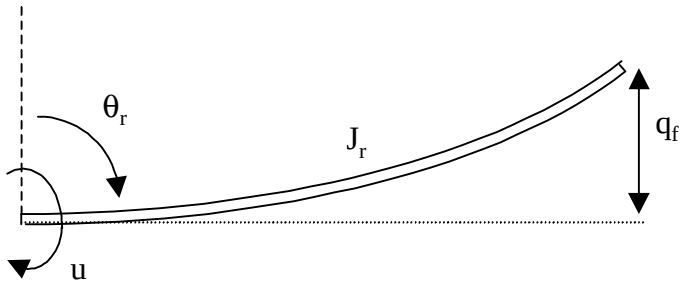


Figure 2.1-12 Step response of motor with fairly rigid shaft

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FLEXIBLE/VIBRATIONAL SYSTEMS

Motor drives with compliant coupling include robotic systems which have flexible joints. Another class of robotic systems are those which have flexible links, such as lightweight arms for fast assembly. In this class are also included many large-scale systems with vibrational modes.



Flexible-Link Pointing System

The mechanical equations of a representative system with one link and one flexible mode are found to be of the form

$$J_r\ddot{q}_r + b_r\dot{q}_r + b(\dot{q}_r - \dot{q}_f) + k(q_r - q_f) = k_r u$$
$$J_f\ddot{q}_f - b(\dot{q}_r - \dot{q}_f) - k(q_r - q_f) = 0 \quad ,$$

where subscript 'r' refers to the rigid dynamics, and subscript 'f' refers to the flexible mode. The rigid mode angle is q_r, the amplitude of the flexible mode is q_f, and the torque input to the link is u(t). Other variables are defined similarly to the case of compliant coupling just discussed. The system is linear. Electrical actuator dynamics are neglected here.

Selecting the state as $x = [x_1\ x_2\ x_3\ x_4]^T = [q_r\ w_r\ q_f\ \dot{q}_f]^T$, with $w = \dot{q}$ the angular velocity, one may write the state equation as

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k}{J_r} & \frac{-(b+b_r)}{J_r} & \frac{k}{J_r} & \frac{b}{J_r} \\ 0 & 0 & 0 & 1 \\ \frac{k}{J_f} & \frac{b}{J_f} & \frac{-k}{J_f} & \frac{-k}{J_f} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{k_r}{J_r} \\ 0 \\ 0 \end{bmatrix} u = Ax + Bu\,.$$

This has the same form as the mechanical subsystem of the motor with compliant coupling.

The output of interest is the rigid rod angle, so that one has the output equation

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x = Cx .$$

If the rod angle and the mode amplitude are both measured, then the output equation is

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x = Cx .$$

The mode amplitude may be measured using, for instance, a strain gauge mounted on the beam.

Analysis and simulation show that this system has significantly different behavior than the flexible-joint case just discussed. In some sense the systems are duals of each other. Note that in the flexible-link case, the input and output are coupled to the same position/velocity state pair, while in the flexible-joint case they are coupled to different position/velocity pairs.

Sample time plots of the motion of the flexible link system are in the figure. Shown are an acceleration/deceleration torque input, the link tip position and velocity, and the amplitudes of the first and second flexible modes.

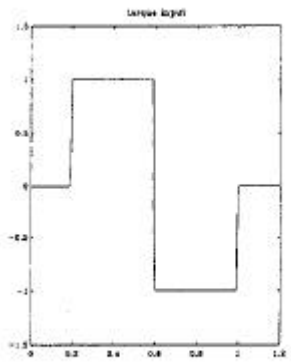


Fig. 2 Acceleration/deceleration torque profile.

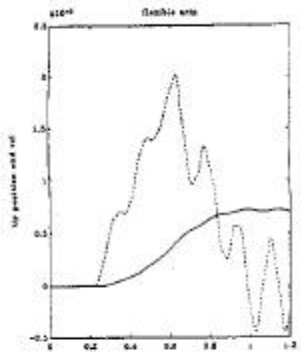


Fig. 3a Open-loop response of flexible arm. Tip position (solid) and vel. (dashed)

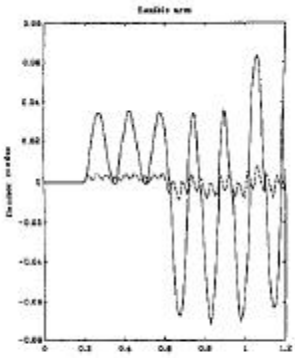


Fig. 3b Open-loop response of flexible arm. Flexible modes.