EE 4314 - Control Systems

Updated: Monday, November 12, 2007

Bode Plot Performance Specifications

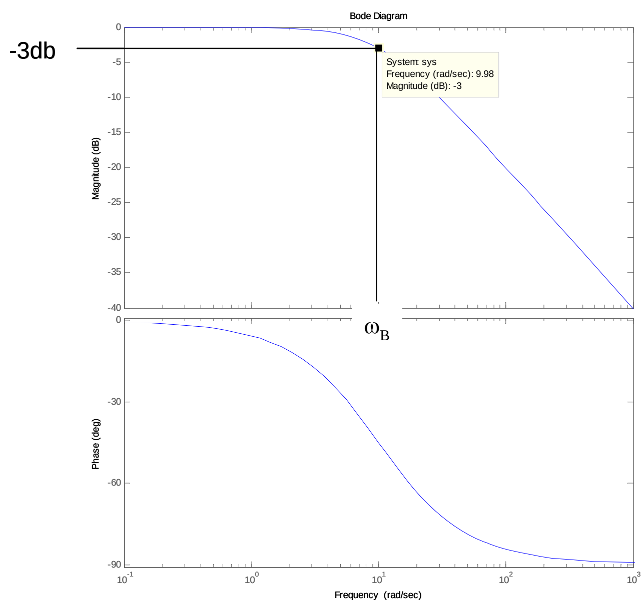
The Bode Plot was developed by Hendrik Wade Bode in 1938 while he worked at Bell Labs. Here we shall show how performance specifications in terms of Bode plots in the frequency domain are related to time domain performance.

Bandwidth and Rise Time

The Bode plot of the transfer function

$$H(s) = \frac{\alpha}{s+\alpha} = \frac{10}{s+10}$$

is shown. The break frequency occurs at 10 rad/sec, the magnitude of the pole.



The *3dB cutoff frequency*, or *bandwidth*, ω_B is the frequency at which the frequency magnitude response has decreased by 3dB from its low frequency value. In this example $\omega_B = \alpha = 10\ rad/s$.

The impulse response of this system is $h(t) = e^{-\alpha t} = e^{-t/\tau}$, where the time constant is $\tau = 1/\alpha = 1/\omega_B$.

The step response rise time is given by $t_r = 2.2\tau$. The settling time is $t_s = 5\tau$.

The time constant is inversely related to the bandwidth. Therefore, as bandwidth increases, the system response becomes faster.

COMPLEX POLE PAIR

A transfer function with a complex pair of poles and no finite zeros can be written as

$$H(s) = \frac{\omega_n^2}{s^2 + 2\alpha s + \omega_n^2} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \equiv \frac{\omega_n^2}{\Delta(s)}.$$

The numerator is chosen to scale the transfer function so that the DC gain (e.g. set $s=0$) is equal to one. The denominator is the *Characteristic polynomial* which can be written in several natural or *canonical forms*, including

$$\Delta(s) = s^2 + 2\alpha s + \omega_n^2 = s^2 + 2\zeta\omega_n s + \omega_n^2.$$

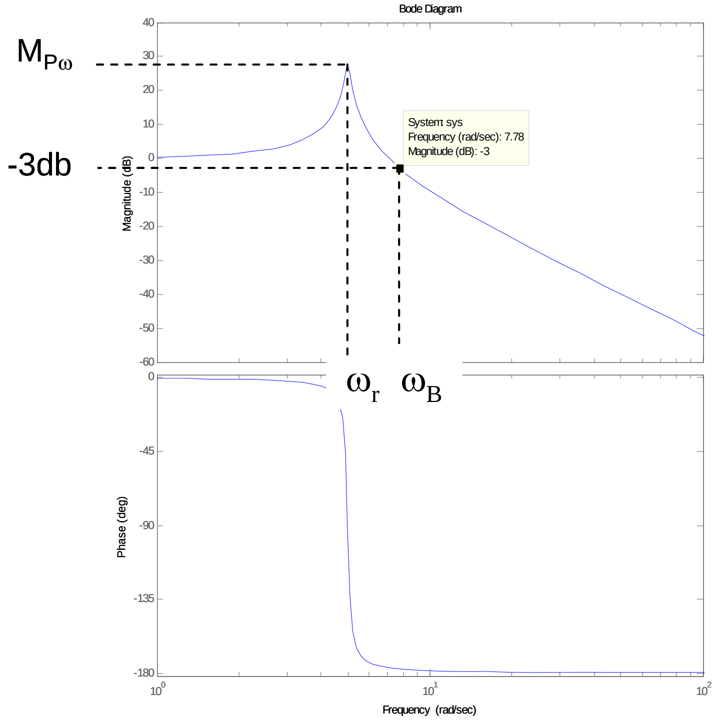
One may also write

$$\Delta(s) = s^2 + 2\alpha s + \omega_n^2 = (s+\alpha)^2 + \beta^2$$

where $\beta^2 + \alpha^2 = \omega_n^2$.

These variables mean something in terms of time domain performance as we have seen. They also mean something in the frequency domain, particularly the damping ratio ζ and the natural frequency ω_n.

The Bode plot for $H(s) = \frac{25}{s^2 + 0.2s + 25}$ is shown.



Recall that for complex poles, the step response is faster than 2.2τ due to the oscillatory components. However, the settling time is $t_s = 5\tau$ and is closely related to the bandwidth; it decreases as bandwidth increases.

The resonant frequency is given for $\zeta \le 0.707$ by

$$\omega_r = \omega_n \sqrt{1-2\zeta^2} .$$

The maximum value of the Bode plot at resonance is given by

$$M_{p\omega} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} .$$

These functions are shown in the figure. From either of these, one may compute the damping ratio and hence the percent overshoot in the time domain.

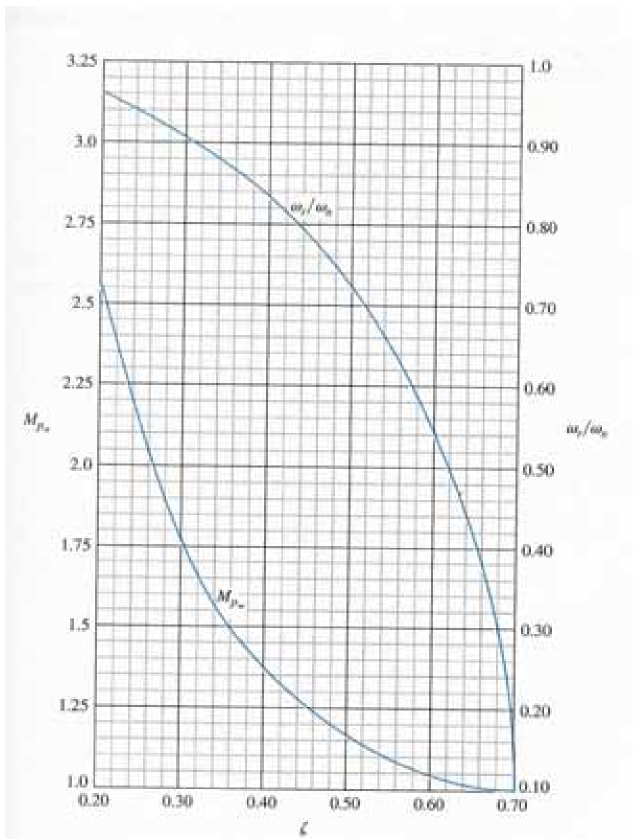


Figure 8.11 from Dorf and Bishop edition 10

The *quality factor*

$$Q = \frac{1}{2\zeta} = \frac{\omega_n}{2\alpha}$$

measures the sharpness of the resonant peak in the Bode plot. Note that this is effectively determined solely by the damping ratio. The poles are complex if *Q> 1/2*.

In terms of the quality factor one may write the characteristic polynomial in the non-dimensional form

$$\Delta(s)=\left(\frac{s}{\omega_n}\right)^2+\frac{1}{Q}\left(\frac{s}{\omega_n}\right)+1$$

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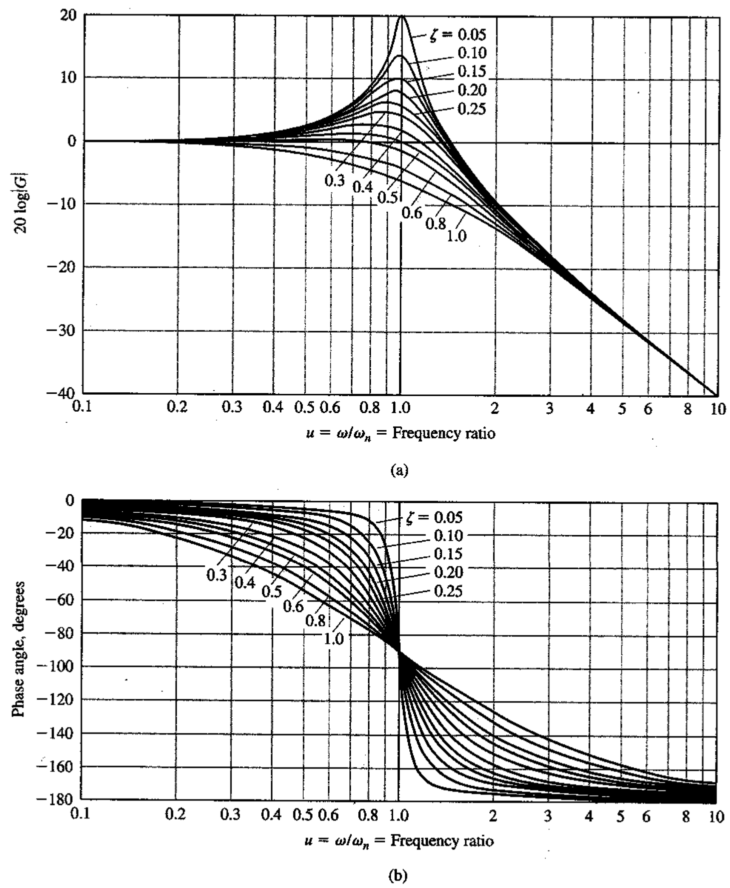
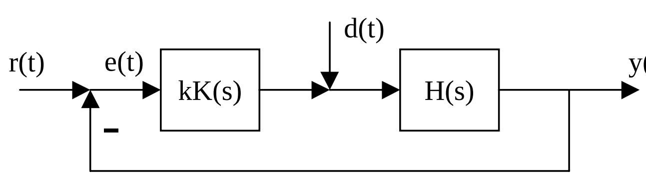


FIGURE 8.10
Bode diagram for $G(j\omega)=[1+(2\zeta/\omega_n)j\omega+(j\omega/\omega_n)^2]^{-1}$.

Figure from Dorf and Bishop, *Modern Control Systems,* edition 8.

Bode Design in terms of the Open-Loop Gain

Consider the tracking controller given in the figure. The plant is *H(s)* and the compensator *K(s);* the feedback gain is k. The function of the tracker is to make the output y(t) follow the command or reference input r(t) by making the tracking error e(t)=r(t)-y(t) small. The disturbance is *d(t).*



The closed-loop transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{kK(s)H(s)}{1 + kK(s)H(s)}.$$

The denominator is

$$\Delta(s) = 1 + kK(s)H(s) \equiv 1 + kG(s)$$

where *G(s)=K(s)H(s)* is the open-loop gain. Note that we use the same symbol for the denominator of T(s) as for the state-variable characteristic polynomial $\Delta(s) = |sI - A|$. However, *1+kG(s)* is actually a polynomial fraction, whose *numerator* is the system characteristic polynomial.

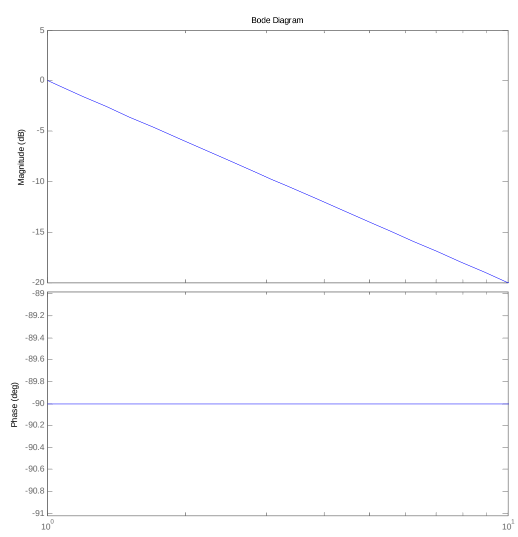
Many design techniques rely on trying to **determine closed-loop properties from open-loop properties**. In root locus design, one uses the open-loop gain *G(s)* to estimate the locations of the closed-loop poles, which are the roots of the numerator of $\Delta(s) = 1 + kG(s)$. The key point of RL design is that it is easier to plot the locations of the closed-loop poles versus the feedback gain parameter *k* than it is to find the actual closed-loop poles themselves. This was extremely important in days before digital computers when finding roots of high-order polynomials was difficult, and it also gives great insight into the properties of the closed-loop system.

Similarly, Bode design uses the Bode plots of the open-loop transfer functions H(s) and K(s)H(s) to select the compensator K(s) to give desirable closed-loop properties including stability, good POV, and fast transient response.

Steady-State Error

Recall that the system is type *N* if there are *N* integrators (i.e. *N* poles at *s=0*) in the feedforward path *K(s)H(s)=G(s).* For zero steady-state error in response to a unit step command, or a unit step disturbance, one requires the system to be of type 1.

The Bode plot of the integrator compensator *K(s)= 1/s* is given in the figure. It has a constant slope of n=1, or -20 dB/decade, and an angle of -90^o. *Therefore, a system of type one has a slope of n=-1 at low frequencies.* To get zero steady-state error in response to a unit step, one must add an integrator to obtain such a slope, unless the Bode plot of H(s) already has this slope at low frequencies.



Crossover Frequency

The crossover frequency ω_c is where the loop gain *G(s)= K(s)H(s)* has a gain of unity, i.e.

$$\left|kG(j\omega_c)\right| = 1 .$$

The closed-loop transfer function is

$$T(s) = \frac{Y(s)}{R(s)} = \frac{kK(s)H(s)}{1 + kK(s)H(s)} .$$

When $\omega << \omega_c$ one has $kG(j\omega) >> 1$ so that

$$\left|T(j\omega)\right| \approx 1$$

and the closed-loop gain is unity.

When $\omega >> \omega_c$ one has $kG(j\omega) << 1$ so that

$$\left|T(j\omega)\right| \approx \left|kG(j\omega)\right|$$

and $|T(j\omega)|$ falls off like $|kG(j\omega)|$.

Therefore, the (closed-loop) bandwidth ω_B is about equal to the (open loop gain) crossover frequency ω_c. To get faster responses, one needs to increase the crossover frequency.

Gain Margin, Phase Margin

The gain margin and phase margin depend on both of the *open-loop gain* Bode plots, magnitude and phase. By extracting information from both plots, GM and PM provide *closed-loop* stability and performance criteria.

The closed-loop denominator is $\Delta(s) = 1 + kK(s)H(s) \equiv 1 + kG(s)$. The closed-loop poles are given by $\Delta(s) = 1 + kK(s)H(s) \equiv 1 + kG(s) = 0$ or $kG(s) = -1$. Therefore, stability may be studied in terms of when $kG(j\omega)$ has a magnitude of one and a phase of 180^o.

The gain margin is the gain increase required to make $|kG(j\omega)| = 1$ when its phase is -180^o.

The phase margin is the phase shift required to make $phase(kG(j\omega)) = -180^o$ when $|kG(j\omega)| = 1$.

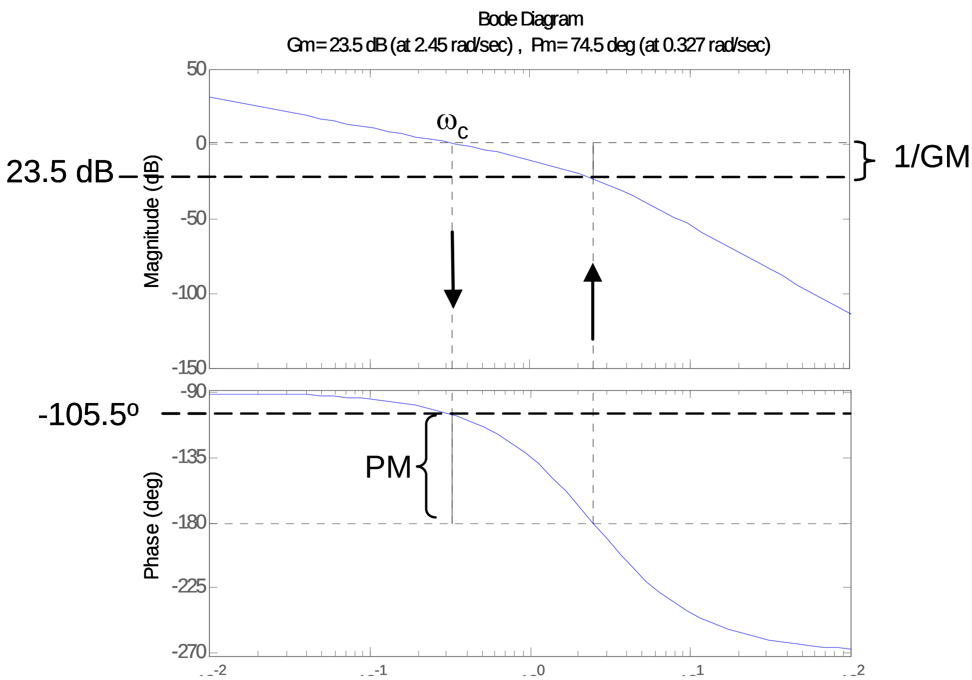
If the magnitude of $kG(j\omega)$ is α when its phase is -180 deg, then the gain margin is $1/\alpha$. The logarithm of $1/\alpha$ is negative the logarithm of α. Therefore, in terms of dB, if α is *d* dB, then $1/\alpha$ is simply –*d* dB.

Note that $|kG(j\omega)| = 1$ means $|kG(j\omega)| = 0\ dB$.

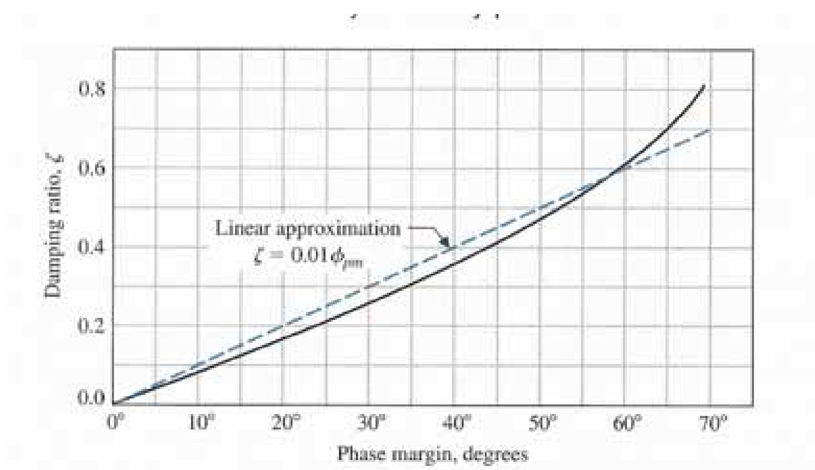
The Bode plot of $kG(s) = \dfrac{2}{s(s+2)(s+3)}$ is shown.

Looking at the crossover frequency ω_c, where the magnitude is equal to one (i.e. 0 dB), the phase margin = -105.5 – (-180)= 74.5 deg.

Looking at the frequency where the phase is -180 deg, the gain margin is -23.5 dB.



The damping ratio increases with phase margin according to the figure. Therefore, to increase damping ratio, we need to increase phase margin.



Relation between PM and damping ratio for a second order system.
Figure 9.21 from Dorf and Bishop ed. 10.