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REACHABILITY AND OBSERVABILITY

Consider a linear state-space system given by

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

with $x(t) \in R^n$ the internal state, $u(t) \in R^m$ the control input, and $y(t) \in R^p$ the measured output. The transfer function is given by

$$H(s) = C(sI - A)^{-1}B + D \, .$$

The *input-decoupling zeros* are those values of *s* for which the $n \times (n+m)$ input-coupling matrix

$$P_I(s) = \begin{bmatrix} sI - A & B \end{bmatrix}$$

loses rank, i.e. has rank less than *n*. Note that this matrix can lose rank only where $(sI - A)$ loses rank, so the input-decoupling zeros must be a subset of the poles.

System *(A,B)* has no input decoupling zeros if and only if

$$rank[(sI - A)^{-1}B] = n$$

over the complex numbers. That is, there exists no nonzero n-vector *w* such that

$$w^T(sI - A)^{-1}B = 0 \quad \textit{for all } s$$

Note that this is the right-hand portion of the transfer function. If there are input-decoupling zeros, control effectiveness of the system is lost and we cannot fully control the system with the given inputs. We should design systems with no input-coupling zeros, i.e. with a fully effective set of inputs.

The *output-decoupling zeros* are those values of *s* for which the $(n+p) \times n$ output-coupling matrix

$$P_O(s) = \begin{bmatrix} sI - A \\ -C \end{bmatrix}$$

loses rank, i.e. has rank less than *n*. Note that this matrix can lose rank only where $(sI - A)$ loses rank, so the output-decoupling zeros must be a subset of the poles.

System *(A,C)* has no output decoupling zeros if and only if

$$rank[C(sI - A)^{-1}] = n$$

over the complex numbers. That is, there exists no nonzero n-vector *v* such that

$$C(sI - A)^{-1}v = 0 \quad \textit{for all } s$$

Note that this is the left-hand portion of the transfer function. If there are output-decoupling zeros, output measurement effectiveness of the system is lost and we cannot observe the full internal state behavior with the given outputs. We should design systems with no output-coupling zeros.

Reachability

For either continuous-time systems or discrete-time systems, the system (A, B, C) is called *reachable* if the control input can be selected to drive any initial state to any desired final state at some final time. This can be done if the input coupling in the system is sufficiently strong, which depends on the input-coupling matrix pair *(A,B)*. Reachability greatly facilitates control systems design. If a system is not reachable, it can be made so by adding additional control inputs.

We can examine the discrete-time expanded state equation to determine reachability conditions for DT systems. It is seen that the DT system is reachable if and only if the *reachability matrix*

$$U = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}.$$

has full rank *n*. Note that *U* is an $n \times nm$ matrix so that it has more columns than rows if $m>1$. Such a matrix is called a flat matrix. (A matrix with more rows than columns is called a sharp matrix.).

In fact, this rank condition on *U* is also necessary and sufficient for reachability of continuous-time systems.

Reachability is equivalent to the absence of input-decoupling zeros. To understand this connection, note that one can write

$$(sI - A)^{-1}B = Bs^{-1} + ABs^{-2} + A^2Bs^{-3} + ...$$

To avoid investigating all powers of *A*, one may use the *Cayley Hamilton Theorem.* This theorem states that

$$\Delta(A) = 0,$$

that is, a matrix satisfies its own characteristic equation. If the characteristic equation is

$$\Delta(s) = s^n + a_1s^{n-1} + ... + a_n,$$

then replace all occurrences of *s* by *A* to obtain (note the last term is a_ns^0)

$$\Delta(A) = A^n + a_1A^{n-1} + ... + a_nI.$$

This is a matrix polynomial. The Cayley-Hamilton Theorem says that

$$A^n = -a_1 A^{n-1} - ... - a_n I ,$$

which states that A^n can be expressed as a linear combination of lower powers of *A*.

Therefore, in the infinite series expansion above, one may stop at $A^{n-1}Bs^{-n}$. Write

$$Bs^{-1} + ABs^{-2} + A^2Bs^{-3} + ... + A^{n-1}Bs^{-n} = \begin{bmatrix} B & AB & A^2B & ... & A^{n-1}B \end{bmatrix} \begin{bmatrix} s^{-1} \\ s^{-2} \\ \vdots \\ s^{-n} \end{bmatrix}$$

which provides a direct relation between the input-coupling term and the reachability matrix *U*. Now, it can be shown that the system is reachable iff there are no input-decoupling zeros.

The reachability matrix is an $n \times nm$ matrix. If there is only one control input (the single-input (SI) case, where m=1), then *U* is square. In this case, it is easy to test whether *U* has rank *n* by making sure the determinant $|U|$ is nonzero. If *m>1* one must find *n* linearly independent columns of *U*, which may be difficult particularly if the number of inputs *m* is large. In this case, define the *reachability gramian*

$$G = UU^T$$

which is a square $n \times n$ matrix. Then the system is reachable iff $|G| \neq 0$.

Many design techniques (e.g. root locus) rely on trying to **determine closed-loop properties from open-loop properties**. This is exactly the intent of the reachability test, which allows one to determine in terms of the open-loop matrices A and B what can be accomplished in the closed-loop system.

The following conditions for reachability are all equivalent and apply for CT and DT systems:

1. (A,B) is reachable
2. $U = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$ has full rank *n*, i.e. has n linearly independent rows.
3. The rows of $(sI - A)^{-1}B$ are linearly independent over the complex numbers.
4. The rows of $e^{At}B$ are linearly independent over $t \in [0, \infty)$
5. $P_I(s) = \begin{bmatrix} sI - A & B \end{bmatrix}$ has full row rank *n* over the complex numbers *s*.

The equivalence of 2 and 3 depends on the relation

$$(sI - A)^{-1}B = Bs^{-1} + ABs^{-2} + A^2Bs^{-3} + ...$$

To understand the relation between 2 and 4 consider the following [p. 167, C.T. Chen, Introduction to Linear System Theory, 1970]. A time-varying matrix *F(t)* which has continuous derivatives up through order *(n-1)* has linearly independent rows over $t \in [0,\infty)$ if and only if the derivative matrix

$$\begin{bmatrix} F(t) & F^{(1)}(t) & \cdots & F^{(n-1)}(t) \end{bmatrix}$$

has rank *n* for some value of *t*. Here the *i*-th derivative is denoted $F^{(i)}(t)$.

Set $F(t) = e^{At}$ and compute the derivative matrix

$$\begin{bmatrix} e^{At}B & Ae^{At}B & A^2e^{At}B & \cdots & A^{n-1}e^{At}B \end{bmatrix}$$

Now set *t=0* to get the reachability matrix.

Consider condition 5, which is equivalent to the existence of a pole s_0 (the input-decoupling zero) and a nonzero *n*-vector *w* such that

$$w^T P_I(s_0) = w^T \begin{bmatrix} s_0 I - A & B \end{bmatrix} = 0$$

i.e.

$$w^T(s_0 I - A) = 0$$

$$w^T B = 0$$

The first equation says that

$$w^T A = s_0 w^T$$

Therefore

$$w^T B = 0$$

$$w^T AB = s_0 w^T B = 0$$

$$w^T A^2 B = s_0 w^T AB = s_0 w^T B = 0$$

$$\vdots$$

which is equivalent to

$$w^T \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} = 0.$$

Observability

For either continuous-time systems or discrete-time systems, let us define the system (A,B,C) to be *observable* if the state can be reconstructed uniquely given measurements of the output over a time interval $[0,T]$. This can be done if the output coupling in the system is sufficiently strong, which depends on the output-coupling matrix pair *(A,C)*. If a system is not observable, it can be made so by adding additional measurements. It turns out that observability means we can design a stable *observer* to reconstruct the internal states given the available measurements. This is important in communications theory, navigation, and elsewhere.

Observability is equivalent to the absence of output-decoupling zeros. To find a test for observability, note that

$$C(sI - A)^{-1} = Cs^{-1} + CAs^{-2} + CA^2 s^{-3} \ldots$$

According to the Cayley-Hamilton Theorem, A^n is dependent on lower powers of *A*. Therefore, note that

$$Cs^{-1} + CAs^{-2} + CA^2 s^{-3} + ... + CA^{n-1}s^{-n} = \begin{bmatrix} s^{-1} & s^{-2} & ... & s^{-n} \end{bmatrix} \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

We call the matrix

$$V = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

the *observability matrix.* We surmise that that (A,C) is observable if and only if matrix V has full rank n. This can be proved, e.g. for DT systems from the expanded state equation.

The observability matrix V has np rows and n columns, so it is called a *sharp matrix* if $p > 1$, for then it has more rows than columns. (Recall that U is a flat matrix.) If the number of outputs p is one, then V is square. Otherwise, it might be quite difficult to determine if V has n linearly independent rows. Define the *observability gramian*

$$G_o = V^T V$$

which is a square $n \times n$ matrix. This matrix has the same rank as V, but it is easy to determine if it has full rank by simply computing its determinant.

The following conditions for observability are all equivalent and apply for CT and DT systems:

1. (A,C) is observable
2. $V = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$ has full rank *n,* i.e. has n linearly independent columns.
3. The columns of $C(sI - A)^{-1}$ are linearly independent over the complex numbers.
4. The columns of Ce^{At} are linearly independent over $t \in [0, \infty)$
5. $P_O(s) = \begin{bmatrix} sI - A \\ -C \end{bmatrix}$ has full column rank *n* over the complex numbers.

Duality

Given a plant (A,B,C), the plant (A^T, C^T, B^T) is known as the *dual system*. In this system, the effects of the inputs and outputs are effectively interchanged. We shall see that duality provides a relation between many concepts, including, e.g., the reachable canonical form and the observable canonical form block diagrams.

We can find a connection between reachability and observability using duality. To this end, dualize the reachability matrix by writing

$$U^T = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}^T = \begin{bmatrix} B^T \\ (AB)^T \\ (A^2B)^T \\ \vdots \\ (A^{n-1}B)^T \end{bmatrix} = \begin{bmatrix} B^T \\ B^T A^T \\ B^T (A^T)^2 \\ \vdots \\ B^T (A^T)^{n-1} \end{bmatrix}.$$

Now, replace (A,B) by the dual system (A^T, C^T), which yields the observability matrix

$$V = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

Clearly, matrix U, with (A,B) replaced by (A^T, C^T), has full rank if and only matrix V, which is based on (A,C) has full rank. Indeed, it can be shown that this is the case. That is, the system is reachable iff the dual system is observable, and vice versa.