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Output Regulation of Heterogeneous MAS- Reduced-Order Synchronization and Geometry

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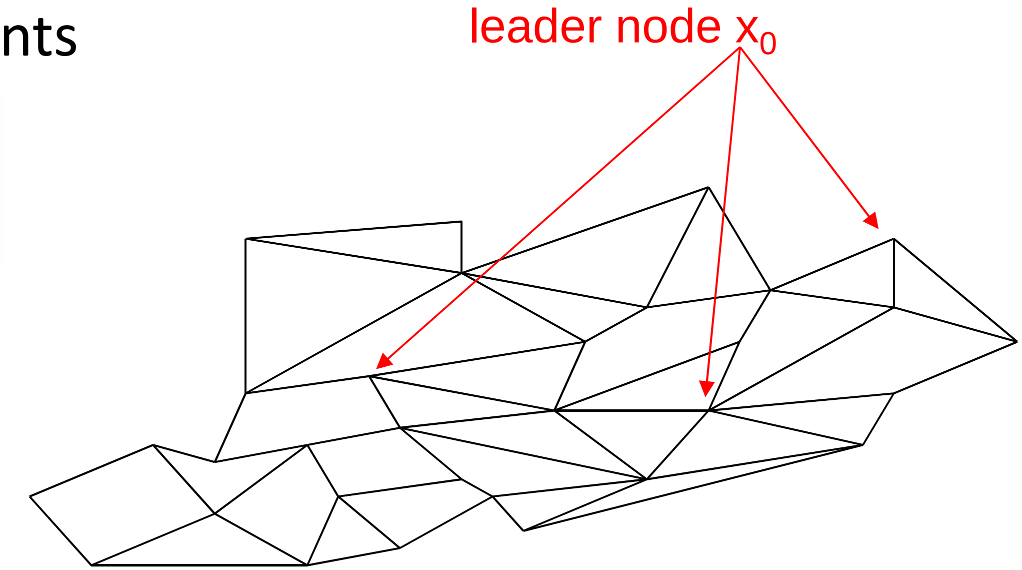
Heterogeneous Multi-Agents

$$\dot{x}_i = A_i x_i + B_i u_i + R_i x_0,$$
$$y_i = C_i x_i,$$



Leader

$$\dot{x}_0 = A_0 x_0,$$
$$y_0 = F x_0,$$

Output regulation error

$$e_i = y_i - y_0 = C_i x_i - F x_0,$$

Design controller such that $e_i \to 0$

Heterogeneous Multi-Agent Systems: Reduced-Order Synchronization and Geometry

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Consider N heterogeneous linear dynamical MAS

$$\dot{x}_i = A_i x_i + B_i u_i, \quad y_i = C_i x_i \tag{1}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{p_i}$, $y_i \in \mathbb{R}^q$, $i = 1, \ldots, N$. The dynamics of a command generator node or leader exosystem are given by

$$\dot{\zeta}_0 = S\zeta_0, \quad y_0 = R\zeta_0 \tag{2}$$

where $\zeta_0(t) \in \mathbb{R}^m$ is the reference state and $y_0(t) \in \mathbb{R}^q$ is the reference output. To avoid triviality assume the leader's dynamics are not asymptotically stable. They may be marginally stable or unstable.

Definition 1: The ***output synchronization problem*** is to design distributed control protocols u_i so that $y_i(t) - y_0(t) \to 0, \ \forall i$.

Heterogeneous Multi-Agents

$$\dot{x}_i = A_i x_i + B_i u_i$$
$$y_i = C_i x_i$$

Leader

$$\dot{\zeta}_0 = S \zeta_0$$
$$y_0 = R \zeta_0$$

Output regulator equations
$$A_i \Pi_i + B_i \Gamma_i = \Pi_i S$$
$$C_i \Pi_i = R$$

Tracking error
$$\bar{\varepsilon}_i = \bar{x}_i - \Pi_i \zeta_0$$

Output regulation error
$$\eta_i(t) = y_i(t) - y_0(t) \to 0$$

Π_i is the insertion map of *S* in *A*

Dynamics are different, state dimensions can be different
o/p reg eqs capture the common core of all the agents dynamics
And define a synchronization manifold

Two Control Methods

Output regulator equations

$$A_i\Pi_i + B_i\Gamma_i = \Pi_i S$$

$$C_i\Pi_i = R$$

Control Method #1

$$\dot{\zeta}_i = S\zeta_i + c\left[\sum_{j=1}^{N} a_{ij}(\zeta_j - \zeta_i) + g_i(\zeta_0 - \zeta_i)\right]$$

$$u_i = K_{1i}(x_i - \Pi_i\zeta_i) + \Gamma_i\zeta_i = K_{1i}x_i + (\Gamma_i - K_{1i}\Pi_i)\zeta_i \equiv K_{1i}x_i + K_{2i}\zeta_i$$

Control Method #2- more intriguing

Local neighborhood output tracking error

$$e_{y_i} \equiv \sum_{j\in N_i} a_{ij}\left(y_j - y_i\right) + g_i\left(y_0 - y_i\right).$$

compensator

$$\begin{cases} \dot{z}_i = F_i z_i + G_i e_{y_i} \\ u_i = K_i x_i + H_i z_i \end{cases}$$

either

$$u_i = K_i x_i + H_i z_i = K_i x_i + (\Gamma_i - K_i\Pi_i)z_i$$

Must know agent and leader's dynamics S,R

Or, assume p-copy in compensator
Then K_i, H_i are independent

Local compensator at each agent- based on leader's dynamics *S.*
Related to p-copy

(1) and the leader's dynamics (2). Therefore, for each agent introduce the dynamics $\dot{\zeta}_i = S\zeta_i$, $z_i = R\zeta_i$ $i = 1, \dots, N$ where $\zeta_i \in \mathbb{R}^m$ and $z_i \in \mathbb{R}^q$. Then the extended dynamical system for agent i is

$$\begin{bmatrix} \dot{x}_i \\ \dot{\zeta}_i \end{bmatrix} = \begin{bmatrix} A_i & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} x_i \\ \zeta_i \end{bmatrix} + \begin{bmatrix} B_i \\ 0 \end{bmatrix} u_i \equiv \tilde{A}_i \tilde{x}_i + \tilde{B}_i u_i \qquad (6)$$

$$\eta_i = [C_i \quad -R] \begin{bmatrix} x_i \\ \zeta_i \end{bmatrix} \equiv \tilde{C}_i \tilde{x}_i \qquad (7)$$

where $\tilde{x} = [x_i^T, \zeta_i^T]^T \in \mathbb{R}^{n_i \times m}$ and $\eta = y_i - z_i \in \mathbb{R}^q$. This is not the

A. State-Space Transformation to Kalman Observable Form

Define O_i as the observability matrix corresponding to the extended system (6), (7) (see [1], [14])

$$O_i = \begin{bmatrix} C_i & -R \\ C_i A_i & -RS \\ \vdots & \vdots \\ C_i A_i^{n_i+m-1} & -RS^{n_i+m-1} \end{bmatrix}. \tag{8}$$

Define q_i as the dimension of the unobservable subspace of (6) and let $\Theta_i \in \mathbb{R}^{(n_i+m)\times q_i}$ be a basis for it so that

$$O_i \Theta_i \equiv O_i \begin{bmatrix} X_{ui} \\ Z_{ui} \end{bmatrix} = 0 \tag{9}$$

where $X_{ui} \in \mathbb{R}^{n_i \times q_i}$ and $Z_{ui} \in \mathbb{R}^{m \times q_i}$. Because of Assumption 1, Z_{ui} and X_{ui} have full column rank [1]. Choose orthonormal bases so that $X_{ui}^T X_{ui} = I$, $Z_{ui}^T Z_{ui} = I$. Let $r_i = m - q_i$ be the dimension of the observable subspace of (6), (7). Choose an orthonormal complementary basis $Z_{oi} \in \mathbb{R}^{m \times r_i}$ so that $z_i = [Z_{oi}, Z_{ui}] \in \mathbb{R}^{m \times m}$ is an orthogonal nonsingular matrix. Next choose $X_{oi} \in \mathbb{R}^{n_i \times r_i}$ to construct $X_i = [X_{oi}, X_{ui}] \in \mathbb{R}^{n_i \times m}$. X_{oi} is a design parameter that is chosen in a certain manner in Section IV. See Lemma 1 below.

Remark 1: This theorem uses a state-space transformation to reveal the complete observability properties of (A_i, B_i, C_i) relative to (S, R) as will be required in Section IV. It is inspired by the innovative ideas

Theorem 1—Transformation to Kalman Observable Form: Consider the extended systems (6), (7) under Assumption 1. Apply the state-space transformation (SST)

$$T_i = \begin{bmatrix} I_{n_i} & -X_i Z_i^{-1} \\ 0 & Z_i^{-1} \end{bmatrix} = \begin{bmatrix} I_{n_i} & -\left(X_{oi} Z_{oi}^T + X_{ui} Z_{ui}^T\right) \\ 0 & Z_{oi}^T \\ 0 & Z_{ui}^T \end{bmatrix} \tag{10}$$

$$\bar{\tilde{x}}_i = T_i \tilde{x}_i = T_i \begin{bmatrix} x_i \\ \zeta_i \end{bmatrix} = \begin{bmatrix} \bar{x}_i \\ \bar{\zeta}_i \end{bmatrix} \equiv \begin{bmatrix} \bar{x}_i \\ \bar{\zeta}_{oi} \\ \bar{\zeta}_{ui} \end{bmatrix} \tag{11}$$

where $\bar{x}_i = x_i - X_i Z_i^T \zeta_i, \bar{\zeta}_{oi} = Z_{oi}^T \zeta_i, \bar{\zeta}_{ui} = Z_{ui}^T \zeta_i$. Then in the new coordinates the dynamics are $\dot{\bar{\tilde{x}}}_i \equiv \bar{A}_i \bar{\tilde{x}}_i + \bar{B}_i u_i = T_i \tilde{A}_i T_i^{-1} \bar{\tilde{x}}_i + T_i \tilde{B}_i u_i, \eta_i = \tilde{C}_i T_i^{-1} \bar{\tilde{x}}_i \equiv \bar{C}_i \bar{\tilde{x}}_i$ with

Transformed Dynamics of Agents plus Local compensator

$$\begin{bmatrix} \dot{x}_i \\ \dot{\zeta}_i \end{bmatrix} = \begin{bmatrix} A_i & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} x_i \\ \zeta_i \end{bmatrix} + \begin{bmatrix} B_i \\ 0 \end{bmatrix} u_i$$

$$\eta_i = [C_i \quad -R] \begin{bmatrix} x_i \\ \zeta_i \end{bmatrix} \equiv \tilde{C}_i \tilde{x}_i$$

$$\begin{bmatrix} \dot{\bar{x}}_i \\ \dot{\bar{\zeta}}_i \end{bmatrix} = \begin{bmatrix} \dot{\bar{x}}_i \\ \dot{\bar{\zeta}}_{oi} \\ \dot{\bar{\zeta}}_{ui} \end{bmatrix} = \begin{bmatrix} A_i & A_{1i} & 0 \\ 0 & S_{oi} & 0 \\ 0 & S_{uoi} & U_i \end{bmatrix} \begin{bmatrix} \bar{x}_i \\ \bar{\zeta}_{oi} \\ \bar{\zeta}_{ui} \end{bmatrix} + \begin{bmatrix} B_i \\ 0 \\ 0 \end{bmatrix} u_i \qquad (12)$$

$$\eta_i = [C_i \; C_{1i} \; 0] \begin{bmatrix} \bar{x}_i \\ \bar{\zeta}_{oi} \\ \bar{\zeta}_{ui} \end{bmatrix} \qquad (13)$$

where $A_{1i} = A_i X_{oi} - X_i Z_i^T S Z_{oi}$, $S_{oi} = Z_{oi}^T S Z_{oi}$, $S_{uoi} = Z_{ui}^T S Z_{oi}$, $U_i = Z_{ui}^T S Z_{ui}$, and $C_{1i} = C_i X_{oi} - R Z_{oi}$. Moreover $\begin{bmatrix} A_i & A_{1i} \\ 0 & S_{oi} \end{bmatrix}$, $[C_i \; C_{1i}]$ is observable. Finally $U_i \equiv S_{ui} = Z_{ui}^T S Z_{ui}$ satisfies

$$A_i X_{ui} = X_{ui} U_i, \quad S Z_{ui} = Z_{ui} U_i. \qquad (4)$$

Based on Theorem 1 the transformed leader's matrix is

$$\bar{S}_i \equiv \begin{bmatrix} S_{oi} & 0 \\ S_{uoi} & U_i \end{bmatrix} = Z_i^T S Z_i. \qquad (15)$$

B. *Intersection of Agent Dynamics and Leader Dynamics*

Invoke a second State-spaceTransformation

dynamics (A_i, B_i, C_i). Given matrix $X_{ui} \in \mathbb{R}^{n_i \times q_i}$ defined in (9), define $\hat{r}_i = n_i - q_i$ and full column rank matrix $\hat{X}_{oi} \in \mathbb{R}^{n_i \times \hat{r}_i}$ to construct $\hat{X}_i = [\hat{X}_{oi}, \hat{X}_{ui}] \in \mathbb{R}^{n_i \times n_i}$ to be nonsingular. Select $X_{ui}, \hat{X}_{oi}$ orthonormal and so that matrix $\hat{X}_i$ is orthogonal.

$$\dot{x}_i = A_i x_i + B_i u_i, \quad y_i = C_i x_i$$

Lemma 2: Define the SST $T_{2i} = [\hat{X}_{oi} \ X_{ui}]^T$. Then in the new coordinates

$$\hat{x}_i = \begin{bmatrix} \hat{x}_{oi} \\ \hat{x}_{ui} \end{bmatrix} = T_{2i} x_i = \begin{bmatrix} \hat{X}_{oi}^T \\ X_{ui}^T \end{bmatrix} x_i \tag{16}$$

the system dynamics (1) have the form

$$\begin{bmatrix} \dot{\hat{x}}_{oi} \\ \dot{\hat{x}}_{ui} \end{bmatrix} = \begin{bmatrix} \hat{A}_{11i} & 0 \\ \hat{A}_{21i} & U_i \end{bmatrix} \begin{bmatrix} \hat{x}_{oi} \\ \hat{x}_{ui} \end{bmatrix} + \begin{bmatrix} \hat{X}_{oi}^T \\ X_{ui}^T \end{bmatrix} B_i u_i \tag{17}$$

$$y_i = [C_i \hat{X}_{oi} \quad C_i X_{ui}] \hat{x}_i. \tag{18}$$

Do not need full compensator

$$\begin{bmatrix} \dot{x}_i \\ \dot{\zeta}_i \end{bmatrix} = \begin{bmatrix} A_i & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} x_i \\ \zeta_i \end{bmatrix} + \begin{bmatrix} B_i \\ 0 \end{bmatrix} u_i$$

$$\eta_i = [C_i \quad -R] \begin{bmatrix} x_i \\ \zeta_i \end{bmatrix} \equiv \tilde{C}_i \tilde{x}_i$$

C. *Reduced-Order Compensator*

Introduce $\bar{\zeta}_{oi} = Z_{oi}^T \zeta_i \in \mathbb{R}^{r_i}$ defined in Theorem 1 as a compensator state at each agent. Define the extended dynamical system with reduced order compensator

$$\begin{bmatrix} \dot{x}_i \\ \dot{\bar{\zeta}}_{oi} \end{bmatrix} = \begin{bmatrix} A_i & 0 \\ 0 & S_{oi} \end{bmatrix} \begin{bmatrix} x_i \\ \bar{\zeta}_{oi} \end{bmatrix} + \begin{bmatrix} B_i \\ 0 \end{bmatrix} u_i. \qquad (19)$$

Lemma 3: Under Assumption 1, (19) can be brought by state-space transformation to the form

$$\begin{bmatrix} \dot{\hat{x}}_{oi} \\ \dot{\hat{x}}_{ui} \\ \dot{\bar{\zeta}}_{oi} \end{bmatrix} = \begin{bmatrix} \hat{A}_{11i} & 0 & 0 \\ \hat{A}_{21i} & U_i & S_{uoi} \\ 0 & 0 & S_{oi} \end{bmatrix} \begin{bmatrix} \hat{x}_{oi} \\ \hat{x}_{ui} \\ \bar{\zeta}_{oi} \end{bmatrix} + \begin{bmatrix} \hat{X}_{oi}^T \\ X_{ui}^T \\ 0 \end{bmatrix} B_i u_i \qquad (20)$$

$$\eta_i = [C_i \hat{X}_{oi} \quad C_i X_{ui} \quad C_{1i}] \begin{bmatrix} \hat{x}_{oi} \\ \hat{x}_{ui} \\ \bar{\zeta}_{oi} \end{bmatrix}. \qquad (21)$$

D. Transformation of Output Regulator Equations

$$A_i \Pi_i + B_i \Gamma_i = \Pi_i S$$
$$C_i \Pi_i = R$$

$$\begin{bmatrix} \dot{x}_i \\ \dot{\zeta}_i \end{bmatrix} = \begin{bmatrix} A_i & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} x_i \\ \zeta_i \end{bmatrix} + \begin{bmatrix} B_i \\ 0 \end{bmatrix} u_i \equiv \tilde{A}_i \tilde{x}_i + \tilde{B}_i u_i \qquad (6)$$

$$\eta_i = [C_i \quad -R] \begin{bmatrix} x_i \\ \zeta_i \end{bmatrix} \equiv \tilde{C}_i \tilde{x}_i \qquad (7)$$

O/p reg. equations (3), (4) are written with respect to (6), (7) as

$$\tilde{A}_i \begin{bmatrix} \Pi_i \\ I \end{bmatrix} + \tilde{B}_i \Gamma_i = \begin{bmatrix} \Pi_i \\ I \end{bmatrix} S, \tilde{C}_i \begin{bmatrix} \Pi_i \\ I \end{bmatrix} = 0. \qquad (23)$$

$$A_i\Pi_i + B_i\Gamma_i = \Pi_i S$$
$$C_i\Pi_i = R$$

Theorem 2—O/p Reg. Equations in Kalman Observable Form: Assume there exist solutions Π_i, Γ_i to the o/p reg. equations (3), (4) and make Assumption 1. Then in the coordinates defined in Theorem 1, the o/p reg. equations (3), (4) are written as

$$A_i\tilde{\Pi}_i + B_i\bar{\Gamma}_i + [A_{1i}\ 0] = \tilde{\Pi}_i \begin{bmatrix} S_{oi} & 0 \\ S_{uoi} & U_i \end{bmatrix} \tag{24}$$

$$C_i\tilde{\Pi}_i + [C_{1i} \quad 0] = 0 \tag{25}$$

where $\bar{\Gamma}_i = \Gamma_i[Z_{\text{oi}}\ Z_{ui}] = \Gamma_i Z_i$ and $\tilde{\Pi}_i = \Pi_i Z_i - X_i = \Pi_i[Z_{\text{oi}}\ Z_{ui}] - [X_{\text{oi}}\ X_{ui}]$.

IV. Reduced-Order Synchronization of Heterogeneous MAS

Definition 2: Define

$$\hat{\zeta}_i = \begin{bmatrix} \bar{\zeta}_{oi} \\ \hat{x}_{ui} \end{bmatrix} \in \mathbb{R}^m, \quad \hat{\bar{x}}_i^0 = x_i - X_i \hat{\zeta}_i. \tag{26}$$

Define the generalized synchronization and tracking errors

$$\hat{\delta} = Z_i \hat{\zeta}_i - Z_0 \bar{\zeta}_0 = Z_i \hat{\zeta}_i - \zeta_0 \in \mathbb{R}^m \tag{27}$$

$$\hat{\bar{\varepsilon}}_i = \hat{\bar{x}}_i^0 - \tilde{\Pi}_i \hat{\zeta}_i \tag{28}$$

where Π_i is the solution to the o/p reg. equations (3) and $\tilde{\Pi}_i = \Pi_i Z_i - X_i$ is the solution to o/p reg. equations (24).

Definition 3—Information Passed Between Agents: Define the information vector passed by agent i to its out-neighbors

$$\tilde{\zeta} = Z_i \hat{\zeta}_i \in \mathbb{R}^m. \tag{29}$$

Design Procedure 1: Design a matrix $F = T^{-1}P$ where P is the unique positive definite solution of the control algebraic Riccati equation (ARE)

$$0 = S^T P + PS + Q - PT^{-1}P \tag{30}$$

with $Q = Q^T \in \mathbb{R}^{m\times m}$, $T = T^T \in \mathbb{R}^{m\times m}$ positive definite.

Design Procedure 2: Design a matrix $F_i = R_i^{-1}B_i^T P_i$ where P_i is the unique positive definite solution of the ARE

$$0 = A_i^T P_i + P_i A_i + Q_i - P_i B_i R_i^{-1} B_i^T P_i \tag{31}$$

with $Q_i = Q_i^T$, $R_i = R_i^T$ positive definite.

Theorem 3—Reduced-Order Synchronizer: Given the agent dynamics (1), assume there exist solutions (Π_i, Γ_i), $\forall i$ to the o/p regulator equations (3), (4) and make Assumption 1. Endow agent i with the dynamic synchronizer of order r_i

$$\dot{\bar{\zeta}}_{oi} = S_{oi}\bar{\zeta}_{oi} + cZ_{oi}^T F \left(\sum_{j=1}^{N} a_{ij}(\tilde{\zeta}_j - \tilde{\zeta}_i) - g_i(\tilde{\zeta}_i - \zeta_0) \right) \quad (32)$$

with F selected as in Design Procedure 1 and $c > 1/(2 \min \operatorname{Re}\{\lambda_l\})$ with $\lambda_l \in spectrum(L+G)$. Assume the graph has a directed spanning tree and $g_i > 0$ for a root node i. Select the control protocol $u_i = u_{1i} + u_{2i}$ with

$$u_{1i} = c\, F_i X_i Z_i^T \left(\sum_{j=1}^{N} a_{ij}(\tilde{\zeta}_j - \tilde{\zeta}_i) - g_i(\tilde{\zeta}_i - \zeta_0) \right) \quad (33)$$

$$u_{2i} = K_i \hat{\bar{\varepsilon}}_i + \bar{\Gamma}_i \hat{\zeta}_i \quad (34)$$

where F_i are selected using Design Procedure 2, K_i are chosen to stabilize $A_i + B_i K_i$, and $\bar{\Gamma}_i = \Gamma_i Z_i$. Assume that $\hat{A}_{21i} = 0$ and

$$X_{ui}^T B_i F_i X_i Z_i^T = Z_{ui}^T F. \quad (35)$$

$$\hat{\zeta}_i = \begin{bmatrix} \bar{\zeta}_{oi} \\ \hat{x}_{ui} \end{bmatrix} \in \mathbb{R}^m,$$

$$\tilde{\zeta} = Z_i \hat{\zeta}_i \in \mathbb{R}^m.$$

$$\hat{\delta}_i = Z_i \hat{\zeta}_i - \zeta_0 \to 0$$

$$\hat{\bar{\varepsilon}}_i = \hat{\bar{x}}_i^0 - \tilde{\Pi}_i \hat{\zeta}_i \to 0$$

$$y_i(t) - y_0(t) \to 0.$$

Thus, synchronization can be achieved if each agent has a mix of a dynamic synchronizer (32) of order r_i and a static state feedback synchronizer (33) of order q_i. The total order of these mechanisms is the dimension m of the state space of the leader's dynamics.

$$\hat{\zeta}_i = \begin{bmatrix} \bar{\zeta}_{oi} \\ \hat{x}_{ui} \end{bmatrix} \in \mathbb{R}^m, \qquad \tilde{\zeta} = Z_i \hat{\zeta}_i \in \mathbb{R}^m.$$

$$\dot{\bar{\zeta}}_{oi} = S_{oi}\bar{\zeta}_{oi} + cZ_{oi}^T F \left(\sum_{j=1}^{N} a_{ij}(\tilde{\zeta}_j - \tilde{\zeta}_i) - g_i(\tilde{\zeta}_i - \zeta_0) \right) \tag{32}$$

$$u_{1i} = c\, F_i X_i Z_i^T \left(\sum_{j=1}^{N} a_{ij}(\tilde{\zeta}_j - \tilde{\zeta}_i) - g_i(\tilde{\zeta}_i - \zeta_0) \right) \tag{33}$$

$$u_{2i} = K_i \hat{\bar{\varepsilon}}_i + \bar{\Gamma}_i \hat{\zeta}_i \tag{34}$$

$$u_i = u_{1i} + u_{2i}$$

Two Control Methods

Control Method #1

$$\dot{\zeta}_i = S\zeta_i + c\left[\sum_{j=1}^{N} a_{ij}(\zeta_j - \zeta_i) + g_i(\zeta_0 - \zeta_i)\right]$$

$$u_i = K_{1i}(x_i - \Pi_i\zeta_i) + \Gamma_i\zeta_i = K_{1i}x_i + (\Gamma_i - K_{1i}\Pi_i)\zeta_i \equiv K_{1i}x_i + K_{2i}\zeta_i$$

Have shown that we only need a dynamic synchronizer of order r_i

$$\dot{\tilde{\zeta}}_{oi} = S_{oi}\tilde{\zeta}_{oi} + cZ_{oi}^T F\left(\sum_{j=1}^{N} a_{ij}(\tilde{\zeta}_j - \tilde{\zeta}_i) - g_i(\tilde{\zeta}_i - \zeta_0)\right) \quad (32)$$

Control Method #2-

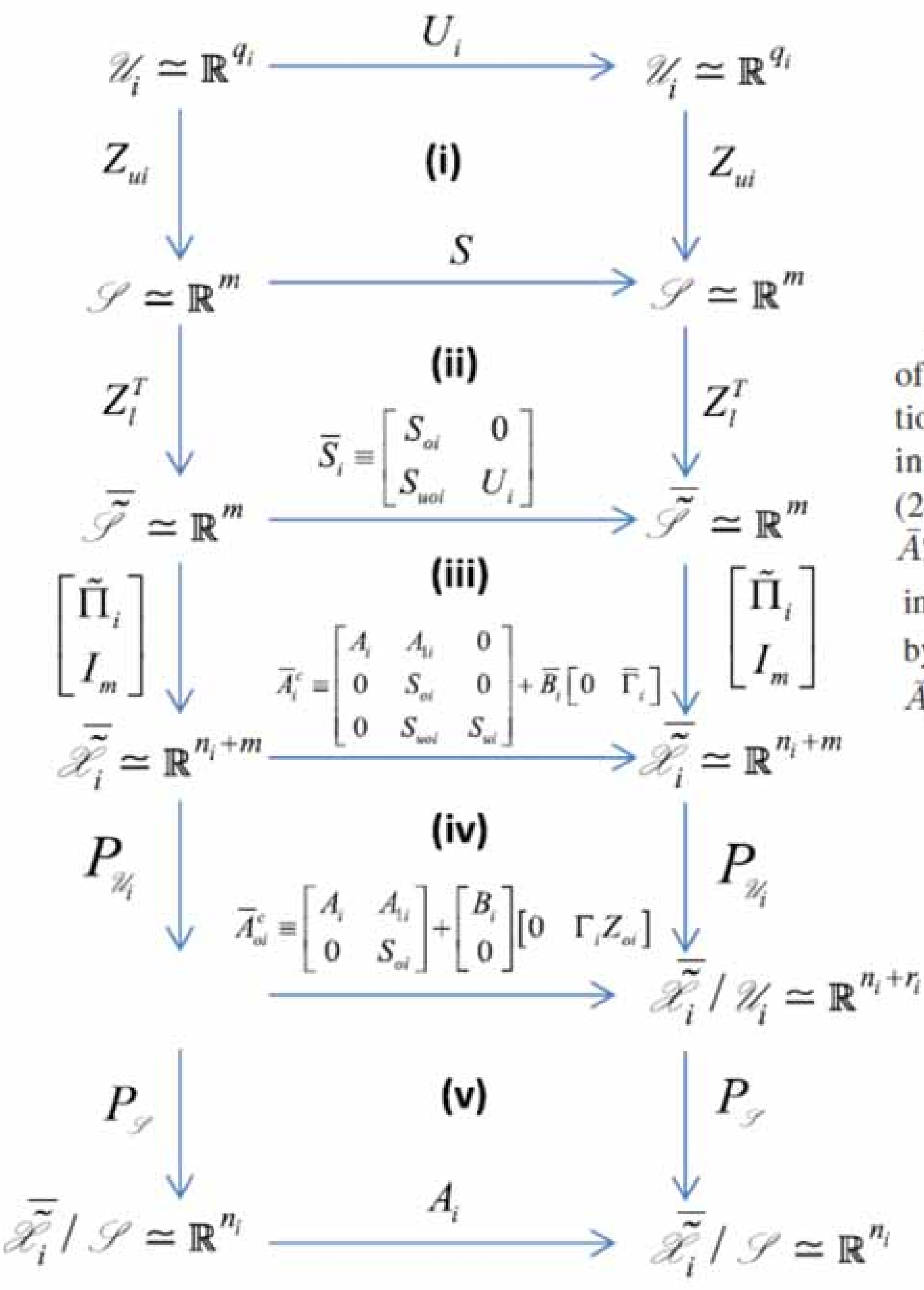
Local neighborhood output tracking error

$$e_{y_i} \equiv \sum_{j\in N_i} a_{ij}\left(y_j - y_i\right) + g_i\left(y_0 - y_i\right).$$

compensator

$$\begin{cases} \dot{z}_i = F_i z_i + G_i e_{y_i} \\ u_i = K_i x_i + H_i z_i \end{cases}$$

Conjecture- do not need a p-copy of *S*. Only need a p-copy of S_{oi}



. 1. Commutative diagram for geometry of heterogeneous cooperative trol.

From Fig. 1 quadrangle (i), $\mathcal{U}_i = \mathrm{Im}(Z_{ui})$ is an invariant subspace of $\mathcal{S} \simeq \mathbb{R}^m$, Z_{ui} is the insertion map of $\mathcal{U}_i$ in $\mathcal{S}$, and U_i is the restriction of S to $\mathcal{U}_i$. Quad (ii) in Fig. 1 is the state-space transformation Z_i^T in Theorem 1. For (iii) The image of the solution to o/p reg. equation (24) $\mathrm{Im}[\tilde{\Pi}_i^T \ I_m]^T$ is an invariant subspace of the closed-loop system $\bar{A}_i^c$ and $[\tilde{\Pi}_i^T \ I_m]^T$ itself is an insertion map. $\bar{S}_i$ is the restriction of $\bar{A}_i$ in to that invariant subspace. On the other hand, A_{oi}^c is the map induced by $\bar{A}_i^c$ in the factor space $\bar{\tilde{\mathcal{X}}}_i/\mathcal{U}_i$, whereas A_i is the map induced by $\bar{A}_i^c$ in the factor space $\bar{\tilde{\mathcal{X}}}_i/\mathcal{S}$.


Joel Santos